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# COMMENSURATORS OF PARABOLIC SUBGROUPS OF COXETER GROUPS

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Abstract. Let $(W,S)$ be a Coxeter system, and let $X$ be a subset of $S$. The subgroup of $W$ generated by $X$ is denoted by $W_X$ and is called a parabolic subgroup. We give the precise definition of the commensurator of a subgroup in a group. In particular, the commensurator of $W_X$ in $W$ is the subgroup of $w$ in $W$ such that $wW_Xw^{-1} \cap W_X$ has finite index in both $W_X$ and $wW_Xw^{-1}$. The subgroup $W_X$ can be decomposed in the form $W_X = W_{X^0} \cdot W_{X^\infty} \simeq W_{X^0} \times W_{X^\infty}$ where $W_{X^0}$ is finite and all the irreducible components of $W_{X^\infty}$ are infinite. Let $Y^\infty$ be the set of $t$ in $S$ such that $m_{s,t} = 2$ for all $s \in X^\infty$. We prove that the commensurator of $W_X$ is $W_{Y^\infty} \cdot W_{X^\infty} \simeq W_{Y^\infty} \times W_{X^\infty}$. In particular, the commensurator of a parabolic subgroup is a parabolic subgroup, and $W_X$ is its own commensurator if and only if $X^0 = Y^\infty$.

## 1. Introduction

Let $S$ be a finite set. A *Coxeter matrix* over $S$ is a matrix $M = (m_{s,t})_{s,t\in S}$ indexed by the elements of $S$ and satisfying

(a) $m_{s,s} = 1$ if $s \in S$,
(b) $m_{s,t} = m_{t,s} \in \{2, 3, 4, \dots, +\infty\}$ if $s, t \in S$ and $s \neq t$.

A Coxeter matrix $M = (m_{s,t})_{s,t\in S}$ is usually represented by its *Coxeter graph* $\Gamma$. This is defined by the following data.

(a) $S$ is the set of vertices of $\Gamma$.
(b) Two vertices $s, t \in S$ are joined by an edge if $m_{s,t} \geq 3$.
(c) The edge joining two vertices $s, t \in S$ is labeled by $m_{s,t}$ if $m_{s,t} \geq 4$.

The *Coxeter system* associated with $M$ (or with $\Gamma$) is the pair $(W,S)$ where $W$ is the group having the presentation

$$W = \langle S \mid (st)^{m_{s,t}} = 1 \text{ if } m_{s,t} < +\infty \rangle \ .$$

The group $W$ is called the *Coxeter group* associated with $M$. Given $X \subseteq S$, we write

$$M_X = (m_{s,t})_{s,t\in X},$$
$\Gamma_X$ the Coxeter graph which represents $M_X$,
$W_X$ the subgroup of $W$ generated by $X$.

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The pair $(W_X, X)$ is the Coxeter system associated with $M_X$ (see [Bo, Ch. IV, §1, $n^o$ 8]). The group $W_X$ is called a *parabolic subgroup* of the Coxeter system $(W, S)$. We assume that the reader is familiar with the theory of Coxeter groups. We refer to [Bo] and [Hu] for general expositions on the subject.

For a group $G$ and for a subgroup $H$ of $G$, we denote by $Z(G)$ the center of $G$, by $Z_G(H)$ the centralizer of $H$ in $G$, by $N_G(H)$ the normalizer of $H$ in $G$, and by $C_G(H)$ the commensurator of $H$ in $G$. Recall that this is defined by

$$C_G(H) = \{g \in G \ ; \ H \cap (gHg^{-1}) \ \text{ has finite index in both } H \text{ and } gHg^{-1}\} \ .$$

Commensurators play an important role in representation theory, especially in the study of induced representations. For example, if a subgroup $H$ of $G$ is its own commensurator, then any finite dimensional irreducible representation of $H$ induces an irreducible representation of $G$ (see [Ma]). If $\mathbf{K}$ is an infinite field and $P$ is a parabolic subgroup of $\mathrm{GL}(n, \mathbf{K})$, then $P$ is its own commensurator (see [BH]). A similar result is obviously not true for Coxeter groups. Indeed, the commensurator of a finite parabolic subgroup is the whole group $W$. However, we prove in this paper that the commensurator of a parabolic subgroup is always a parabolic subgroup (Corollary 2.2), and we give a criterion which decides whether a parabolic subgroup is its own commensurator (Corollary 2.3).

The goal of this paper is to determine the commensurator of a parabolic subgroup $W_X$ of a Coxeter system. This subgroup can be decomposed in the form $W_X = W_{X^0} \cdot W_{X^\infty} \simeq W_{X^0} \times W_{X^\infty}$, where $W_{X^0}$ is finite and all the irreducible components of $W_{X^\infty}$ are infinite. In a first step (Proposition 2.4), we prove that the commensurator of $W_X$ is the normalizer of $W_{X^\infty}$. In a second step (Proposition 2.5), we prove that the normalizer of $W_{X^\infty}$ is $QZ_W(W_{X^\infty}) \cdot W_{X^\infty}$, where

$$QZ_W(W_{X^\infty}) = \{w \in W \ ; \ wX^\infty w^{-1} = X^\infty\}$$

is the *quasi-centralizer* of $W_{X^\infty}$. In a third step (Proposition 2.6), we prove that $QZ_W(W_{X^\infty})$ is $W_{Y^\infty}$, where $Y^\infty$ is the set of $t$ in $S$ such that $m_{s,t} = 2$ for all $s \in X^\infty$. Finally, from Propositions 2.4, 2.5 and 2.6, we deduce the following expression of the commensurator of $W_X$ (Theorem 2.1):

$$C_W(W_X) = W_{Y^\infty} \cdot W_{X^\infty} \simeq W_{Y^\infty} \times W_{X^\infty} \ .$$

We state our results precisely in Section 2, and we prove them in Section 3.

## 2. Statements

From now on, we fix a Coxeter system $(W, S)$.

Let $X$ be a subset of $S$. Let $\Gamma_1, \dots, \Gamma_n$ be the connected components of $\Gamma_X$ and, for $i \in \{1, \dots, n\}$, let $X_i$ be the set of vertices of $\Gamma_i$. The group $W_{X_i}$ is called an *irreducible component* of $W_X$. It is clear that

$$W_X = W_{X_1} \cdot \dots \cdot W_{X_n} \simeq W_{X_1} \times \dots \times W_{X_n} \ .$$

We assume that $W_{X_i}$ is finite if $i = 1, \dots, r$, and that $W_{X_i}$ is infinite if $i = r + 1, \dots, n$. We set

$$X^0 = X_1 \cup \dots \cup X_r \ ,$$
$$X^\infty = X_{r+1} \cup \dots \cup X_n \ .$$

Then

$$W_X = W_{X^0} \cdot W_{X^\infty} \ \simeq W_{X^0} \times W_{X^\infty} \ ,$$

the group $W_{X^0}$ is finite, and all the irreducible components of $W_{X^\infty}$ are infinite.

**Theorem 2.1.** *Let $X$ be a subset of $S$. Then*

$$C_W(W_X) = W_{Y^\infty} \cdot W_{X^\infty} = W_{Y^\infty \cup X^\infty} \simeq W_{Y^\infty} \times W_{X^\infty},$$

*where*

$$Y^\infty = \{t \in S \; ; \; m_{s,t} = 2 \;\; \text{for all } s \in X^\infty\} \; .$$

**Corollary 2.2.** *The commensurator of a parabolic subgroup of $(W,S)$ is a parabolic subgroup.*

**Corollary 2.3.** *Let $X$ be a subset of $S$. Then $W_X$ is its own commensurator if and only if $X^0$ is the set of $t \in S$ such that $m_{s,t} = 2$ for all $s \in X^\infty$.*

Theorem 2.1 is a direct consequence of the following Propositions 2.4, 2.5, and 2.6.

**Proposition 2.4.** *Let $X$ be a subset of $S$. Then the commensurator of $W_X$ in $W$ is equal to the normalizer of $W_{X^\infty}$ in $W$.*

We define the *quasi-center* of $(W,S)$ to be

$$QZ(W,S) = \{w \in W \; ; \; wSw^{-1} = S\} \; .$$

Similarly, we define the *quasi-centralizer* of a parabolic subgroup $W_X$ of $(W,S)$ to be

$$QZ_W(W_X) = \{w \in W \; ; \; wXw^{-1} = X\} \; .$$

**Proposition 2.5.** *Let $X$ be a subset of $S$. Then*

$$N_W(W_X) = QZ_W(W_X) \cdot W_X \; .$$

*Moreover,*

$$QZ_W(W_X) \cap W_X = QZ(W_X, X) \; .$$

**Proposition 2.6.** *Let $X$ be a subset of $S$ such that all the irreducible components of $W_X$ are infinite (i.e. $X = X^\infty$). Let*

$$Y = \{t \in S \; ; \; m_{s,t} = 2 \;\; \text{for all } s \in X\} \; .$$

*Then the quasi-centralizer of $W_X$ is equal to $W_Y$.*

Proposition 2.4 is a consequence of [So, Lemma 2]. Proposition 2.5 is stated in [Ho] for finite type Coxeter systems (see also [Kr, Ch. 3]). Moreover, its proof is quite simple. Proposition 2.6 is a consequence of [De, Prop. 5.5].

## 3. Proofs

First, we state in Lemmas 3.1 and 3.2 some well-known facts that will be required later. Recall that each $w \in W$ can be written $w = s_1 \dots s_r$ where $s_i \in S$ for all $i \in \{1, \dots, r\}$. If $r$ is as small as possible, then $r$ is called the *length* of $w$ and is denoted by $l(w)$.

**Lemma 3.1** (Bourbaki [Bo, Ch. IV, §1, Ex. 3]). *Let $X$ and $X'$ be two subsets of $S$.*

(i) *Let* $w \in W$. *There is a unique element* $v$ *of minimal length in* $W_X w W_{X'}$. *Moreover, each* $w' \in W_X w W_{X'}$ *can be written as* $w' = uvu'$, *where* $u \in W_X$, $u' \in W_{X'}$, *and* $l(w') = l(u) + l(v) + l(u')$. *An element* $v$ *is called* $(X, X')$*-reduced if it is of minimal length in* $W_X v W_{X'}$.

(ii) *If an element* $v$ *is* $(X, \emptyset)$*-reduced, then* $l(uv) = l(u) + l(v)$ *for all* $u \in W_X$.

(iii) *If an element* $v$ *is* $(\emptyset, X')$*-reduced, then* $l(vu') = l(v) + l(u')$ *for all* $u' \in W_{X'}$.

(iv) *An element* $v$ *is* $(X, \emptyset)$*-reduced if and only if* $l(sv) > l(v)$ *for all* $s \in X$.

(v) *An element* $v$ *is* $(\emptyset, X')$*-reduced if and only if* $l(vs') > l(v)$ *for all* $s' \in X'$.

(vi) *An element* $v$ *is* $(X, X')$*-reduced if and only if it is both* $(X, \emptyset)$*-reduced and* $(\emptyset, X')$*-reduced.*

**Lemma 3.2** (Bourbaki [Bo, Ch. IV, §1, Ex. 22])**.** *Let* $w_0$ *be an element of* $W$. *The following statements are equivalent.*

(1) $l(sw_0) < l(w_0)$ *for all* $s \in S$.

(2) $l(w_0 s) < l(w_0)$ *for all* $s \in S$.

(3) $l(ww_0) = l(w_0) - l(w)$ *for all* $w \in W$.

(4) $l(w_0 w) = l(w_0) - l(w)$ *for all* $w \in W$.

*Such an element is unique and exists if and only if* $W$ *is finite. Then it is the unique element of maximal length in W. Moreover,* $w_0^2 = 1$ *and* $w_0 S w_0 = S$.

The following proposition is the key of the proof of Proposition 2.4.

**Proposition 3.3** (Solomon [So, Lemma 2])**.** *Let* $X$ *and* $X'$ *be two subsets of* $S$, *and let* $v$ *be a* $(X, X')$*-reduced element of* $W$. *Then*

$$W_X \cap (v W_{X'} v^{-1}) = W_Y,$$

*where* $Y = (vX'v^{-1}) \cap X$.

**Corollary 3.4.** *Let* $X$ *and* $X'$ *be two subsets of* $S$, *and let* $w$ *be an element of* $W$. *We write* $w = u_0 v u_0'$, *where* $u_0 \in W_X$, $u_0' \in W_{X'}$, *and* $v$ *is* $(X, X')$*-reduced. Then*

$$W_X \cap (w W_{X'} w^{-1}) = u_0 W_Y u_0^{-1},$$

*where* $Y = (vX'v^{-1}) \cap X$.

*Proof.*

$$\begin{aligned} W_X \cap (w W_{X'} w^{-1}) &= W_X \cap (u_0 v u_0' W_{X'} u_0'^{-1} v^{-1} u_0^{-1}) \\ &= W_X \cap (u_0 v W_{X'} v^{-1} u_0^{-1}) \\ &= u_0((u_0^{-1} W_X u_0) \cap (v W_{X'} v^{-1})) u_0^{-1} \\ &= u_0(W_X \cap (v W_{X'} v^{-1})) u_0^{-1} \\ &= u_0 W_Y u_0^{-1}\ . \end{aligned}$$

□

*Proof of Proposition 2.4.* Let $w \in N_W(W_{X^\infty})$. Then

$$W_{X^\infty} = w W_{X^\infty} w^{-1} \subseteq W_X \cap (w W_X w^{-1})\ ,$$

the group $W_{X^\infty}$ has finite index in $W_X$, and the group $w W_{X^\infty} w^{-1}$ has finite index in $w W_X w^{-1}$. Thus $W_X \cap (w W_X w^{-1})$ has finite index in both $W_X$ and $w W_X w^{-1}$. This shows that $N_W(W_{X^\infty}) \subseteq C_W(W_X)$.

Let $w \in C_W(W_X)$. We write $w = u_0 v u_0'$, where $u_0, u_0' \in W_X$ and $v$ is $(X,X)$-reduced. By Corollary 3.4,

$$W_X \cap (wW_Xw^{-1}) = u_0 W_Y u_0^{-1},$$

where $Y = (vXv^{-1}) \cap X$. Let $Y^0 = Y \cap X^0$, and let $Y^\infty = Y \cap X^\infty$. For a group $G$ and for a subgroup $H$ of $G$, we denote by $|G : H|$ the index of $H$ in $G$. Then

$$\begin{aligned} |W_X : W_X \cap (wW_Xw^{-1})| &= |W_X : u_0 W_Y u_0^{-1}| = |W_X : W_Y| \\ &= |W_{X^0} : W_{Y^0}| \cdot |W_{X^\infty} : W_{Y^\infty}| \ . \end{aligned}$$

If $Y^\infty \neq X^\infty$, then, by [De, Prop. 4.2], $W_{Y^\infty}$ has infinite index in $W_{X^\infty}$; thus $W_X \cap (wW_Xw^{-1})$ has infinite index in $W_X$, too. This is not the case; thus $Y^\infty = X^\infty$. Let $\Gamma_1, \dots, \Gamma_n$ be the connected components of $\Gamma_X$, and, for $i = 1, \dots, n$, let $X_i$ be the set of vertices of $\Gamma_i$. We assume that $X^0 = X_1 \cup \dots \cup X_r$ and that $X^\infty = X_{r+1} \cup \dots \cup X_n$. Let $i \in \{r+1, \dots, n\}$. Then

$$v^{-1}X_iv \subseteq v^{-1}X^\infty v = v^{-1}Y^\infty v \subseteq v^{-1}Yv \subseteq X \ .$$

Thus there exists $j \in \{1, \dots, r, r{+}1, \dots, n\}$ such that $v^{-1}X_iv \subseteq X_j$. The group $W_{X_i}$ is infinite and $v^{-1}W_{X_i}v \subseteq W_{X_j}$; thus $W_{X_j}$ is infinite, and so $j \in \{r+1, \dots, n\}$. This shows that $v^{-1}X^\infty v \subseteq X^\infty$; thus $vX^\infty v^{-1} = X^\infty$; therefore $vW_{X^\infty}v^{-1} = W_{X^\infty}$. On the other hand, since $W_X = W_{X^0} \cdot W_{X^\infty} \simeq W_{X^0} \times W_{X^\infty}$, we have $uW_{X^\infty}u^{-1} = W_{X^\infty}$ for all $u \in W_X$. So,

$$wW_{X^\infty}w^{-1} = u_0vu_0'W_{X^\infty}u_0'^{-1}v^{-1}u_0^{-1} = u_0vW_{X^\infty}v^{-1}u_0^{-1} = u_0W_{X^\infty}u_0^{-1} = W_{X^\infty} \ .$$

This shows that $C_W(W_X) \subseteq N_W(W_{X^\infty})$. □

*Proof of Proposition 2.5.* The inclusion

$$QZ_W(W_X) \cdot W_X \subseteq N_W(W_X)$$

is obvious.

Let $w \in N_W(W_X)$. We write $w = vu$, where $u \in W_X$, $v$ is $(\emptyset, X)$-reduced, and $l(w) = l(v) + l(u)$. We have

$$wW_Xw^{-1} = vW_Xv^{-1} = W_X \ .$$

The element $v$ is of minimal length in $vW_X = W_Xv$; thus $v$ is also $(X, \emptyset)$-reduced. If $s \in X$, then, by Lemma 3.1,

$$\begin{aligned} & l(v) + 1 = l(vs) = l(vsv^{-1}v) = l(vsv^{-1}) + l(v) \\ \Rightarrow\ & l(vsv^{-1}) = 1 \\ \Rightarrow\ & vsv^{-1} \in W_X \cap S = X \ . \end{aligned}$$

So, $vXv^{-1} \subseteq X$; thus $vXv^{-1} = X$; therefore $v \in QZ_W(W_X)$. This shows that $N_W(W_X) \subseteq QZ_W(W_X) \cdot W_X$.

The equality

$$QZ_W(W_X) \cap W_X = QZ(W_X, X)$$

is obvious. □

Before proving Proposition 2.6, we recall some facts on root systems. Let $V$ be a real vector space having a basis $\{e_s; s \in S\}$ in one-to-one correspondence with $S$. Let $B$ be the symmetric bilinear form on $V$ defined by

$$B(e_s, e_t) = \begin{cases} -\cos(\pi/m_{s,t}) & \text{if } m_{s,t} < +\infty, \\ -1 & \text{if } m_{s,t} = +\infty. \end{cases}$$

There is an action of $W$ on $V$ defined by

$$s(x) = x - 2B(x, e_s)e_s$$

if $s \in S$ and $x \in V$. This action is called the *canonical representation* of $(W, S)$. The *root system* $\Phi$ of $(W, S)$ is the collection of all vectors $w(e_s)$ where $w \in W$ and $s \in S$. By [Bo, Ch. V, §4, Ex. 8], every root $\alpha$ can be uniquely written in the form

$$\alpha = \sum_{s \in S} a_s e_s \quad (a_s \in \mathbf{R}),$$

where either all $a_s$ are positive, or all $a_s$ are negative. We call $\alpha$ *positive* and write $\alpha > 0$ if $a_s \geq 0$ for all $s \in S$. We call $\alpha$ *negative* and write $\alpha < 0$ if $a_s \leq 0$ for all $s \in S$.

**Proposition 3.5** (Deodhar [De, Prop. 3.1])**.** *Let*

$$T = \{wsw^{-1} \ ; \ w \in W \quad \text{and } s \in S\} \ ,$$

*and let $\Phi^+$ be the set of positive roots. For $\alpha = w(e_s)$, we write $r_\alpha = wsw^{-1}$. Then the function $\Phi^+ \to T$ $(\alpha \mapsto r_\alpha)$ is well-defined and bijective.*

**Proposition 3.6** (Deodhar [De, Prop. 2.2])**.** *Let $w \in W$, and let $s \in S$. Then $l(ws) > l(w)$ if and only if $w(e_s) > 0$.*

For a subset $X$ of $S$, we write

$$E_X = \{e_s \ ; \ s \in X\} \ .$$

The following lemma is an easy consequence of Propositions 3.5 and 3.6.

**Lemma 3.7.** *Let $X$ and $X'$ be two subsets of $S$, and let $w$ be an element of $W$. The following statements are equivalent.*

(1) $w(E_X) = E_{X'}$.
(2) $wXw^{-1} = X'$ *and* $l(ws) > l(w)$ *for all* $s \in X$.

For $X \subseteq S$ such that $W_X$ is finite, we denote by $w_X$ the unique element of maximal length in $W_X$.

Let $X$ be a subset of $S$, and let $t$ be an element of $S \setminus X$. Let $\Gamma_0$ be the connected component of $\Gamma_{\{t\} \cup X}$ containing $t$, and let $Y_0$ be the set of vertices of $\Gamma_0$. We say that $t$ is $X$-*admissible* if $W_{Y_0}$ is finite. In that case, we write

$$c(t, X) = w_{Y_0} w_{X_0},$$

where $X_0 = Y_0 \setminus \{t\}$. It is the element of minimal length in $w_{Y_0} W_X$. In particular, $c(t, X)$ is $(\emptyset, X)$-reduced. By Lemma 3.2 and Lemma 3.7, there exists a subset $X'$ of $\{t\} \cup X$ such that

$$c(t, X)(E_X) = E_{X'} \ .$$

If $X = X^\infty$, then $t$ is $X$-admissible if and only if $m_{s,t} = 2$ for all $s \in X$. In that case, $c(t, X) = t$ and $c(t, X)(E_X) = E_X$.

**Proposition 3.8** (Deodhar [De, Prop. 5.5]). *Let $X$ and $X'$ be two subsets of $S$, and let $w$ be an element of $W$. If $w(E_X) = E_{X'}$, then there exist sequences*

$X_0 = X, X_1, \dots, X_n = X'$ *of subsets of* $S$,
$t_0, t_1, \dots, t_{n-1}$ *of elements of* $S$,

*such that*

(a) $t_i \in S \setminus X_i$ *and* $t_i$ *is* $X_i$*-admissible* $(i = 0, 1, \dots, n-1)$,
(b) $c(t_i, X_i)(E_{X_i}) = E_{X_{i+1}}$ $(i = 0, 1, \dots, n-1)$,
(c) $w = c(t_{n-1}, X_{n-1}) \dots c(t_1, X_1) c(t_0, X_0)$.

The following Lemmas 3.9 and 3.10 are preliminary results to the proof of Proposition 2.6.

**Lemma 3.9.** *Let $X$ and $X'$ be two subsets of $S$, and let $w$ be an element of $W$. If $wXw^{-1} = X'$, then $w$ can be written $w = vu$, where $u \in QZ(W_X, X)$, $vXv^{-1} = X'$, and $l(vs) > l(v)$ for all $s \in X$.*

*Proof.* We write $w = vu$, where $u \in W_X$, $v$ is $(\emptyset, X)$-reduced, and $l(w) = l(v) + l(u)$. We have

$$wW_Xw^{-1} = vW_Xv^{-1} = W_{X'} .$$

The element $v$ is of minimal length in $vW_X = W_{X'}v$; thus $v$ is also $(X', \emptyset)$-reduced. If $s \in X$, then, by Lemma 3.1,

$$\begin{aligned} l(v) + 1 = l(vs) &= l(vsv^{-1}v) = l(vsv^{-1}) + l(v) \\ \Rightarrow l(vsv^{-1}) &= 1 \\ \Rightarrow vsv^{-1} &\in W_{X'} \cap S = X' . \end{aligned}$$

So, $vXv^{-1} \subseteq X'$. Similarly, $v^{-1}X'v \subseteq X$. Thus $vXv^{-1} = X'$.

Since $v$ is $(\emptyset, X)$-reduced, by Lemma 3.1, $l(vs) > l(v)$ for all $s \in X$.

Finally,

$$\begin{aligned} wXw^{-1} &= vuXu^{-1}v^{-1} = X' \\ \Rightarrow uXu^{-1} &= v^{-1}X'v = X. \end{aligned}$$

Thus $u \in QZ(W_X, X)$. □

**Lemma 3.10** (Bourbaki [Bo, Ch. V, §4, Ex. 3]). *We suppose that $(W, S)$ is irreducible.*

(i) *If $W$ is finite, then $QZ(W, S) = \{1, w_0\}$, where $w_0$ is the unique element of maximal length in $W$.*
(ii) *If $W$ is infinite, then $QZ(W, S) = \{1\}$.*

*Proof of Proposition 2.6.* The inclusion

$$W_Y \subseteq QZ_W(W_X)$$

is obvious.

Let $w \in QZ_W(W_X)$. By Lemma 3.9, $w$ can be written $w = vu$, where $u \in QZ(W_X, X)$, $vXv^{-1} = X$, and $l(vs) > l(v)$ for all $s \in X$. Since $X = X^\infty$, by Lemma 3.10, $QZ(W_X, X) = \{1\}$; thus $u = 1$. By Lemma 3.7, $v(E_X) = E_X$. By Proposition 3.8, there exist sequences

$X = X_0, X_1, \dots, X_n = X$ of subsets of $S$,
$t_0, t_1, \dots, t_{n-1}$ of elements of $S$,

such that

(a) $t_i \in S \setminus X_i$ and $t_i$ is $X_i$-admissible $(i = 0, 1, \dots, n-1)$,
(b) $c(t_i, X_i)(E_{X_i}) = E_{X_{i+1}}$ $(i = 0, 1, \dots, n-1)$,
(c) $v = c(t_{n-1}, X_{n-1}) \dots c(t_1, X_1) c(t_0, X_0)$.

Since $X = X^\infty$, if $X_i = X$, then $m_{t_i,s} = 2$ for all $s \in X$ (namely, $t_i \in Y$), $c(t_i, X_i) = t_i$, and $X_{i+1} = X$ (since $t_i(E_X) = E_X$). Since $X_0 = X$, it follows that $c(t_i, X_i) = t_i \in Y$ for all $i = 0, 1, \dots, n-1$. Thus

$$w = v = t_{n-1} \dots t_1 t_0 \in W_Y \ .$$

This shows that $QZ_W(W_X) \subseteq W_Y$. □

## References


[Bo] N. Bourbaki, *"Groupes et algèbres de Lie, Chapitres IV–VI"*, Hermann, Paris, 1968. MR **39:**1590

[Br] K. S. Brown, *"Buildings"*, Springer-Verlag, New York, 1989. MR **90e:**20001

[BH] M. Burger and P. de la Harpe, *Irreducible representations of discrete groups*, in preparation.

[De] V. V. Deodhar, *On the root system of a Coxeter group*, Comm. Algebra **10** (1982), 611-630. MR **83j:**20052a

[Ho] R. B. Howlett, *Normalizers of parabolic subgroups of reflection groups*, J. London Math. Soc. (2) **21** (1980), 62–80. MR **81g:**20094

[Hu] J. E. Humphreys, *"Reflection groups and Coxeter groups"*, Cambridge studies in advanced mathematics, vol. 29, Cambridge University Press, 1990. MR **92h:**20002

[Kr] D. Krammer, *"The conjugacy problem for Coxeter groups"*, Ph. D. Thesis, Utrecht, 1994.

[Ma] G. W. Mackey, *"The theory of unitary group representations"*, The University of Chicago Press, 1976. MR **53:**686

[So] L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), 255–264. MR **56:**3104



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