

NOTE ON CLARK'S THEOREM FOR p -ADIC CONVERGENCE

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ABSTRACT. We must read Clark's statement under the hypothesis that the *negative* of each zero of the indicial polynomial is non-Liouville. In this note we shall give the example for which under the original hypothesis the statement does not hold.

INTRODUCTION

Clark's article [1] is referred to by many authors because he first studied the p -adic convergence of power series solutions at a singular point and pointed out the importance of p -adic non-Liouvilleness of exponents. However, his result is not properly stated. We must read his statement under the hypothesis that the *negative* of each zero of the indicial polynomial is non-Liouville. In this note we shall give the example for which under the original hypothesis the statement does not hold. For another approach to p -adic convergence using the notion of p -adic Liouvilleness in the sense of Schikhof [2], see [3].

Throughout this note \mathbb{C}_p denotes the completion of the algebraic closure of \mathbb{Q}_p , and the p -adic valuation $|\cdot|_p$ is simply written $|\cdot|$. By \mathbb{N} we denote the positive integers, and by \mathbb{N}^0 the non-negative integers. \mathbb{Z} and \mathbb{Z}_p denote the rational integers and the ring of p -adic integers of \mathbb{Q}_p , respectively.

1. SOME DEFINITIONS

The statement in question is Clark's Theorem 3 [1]. We shall give a summary of this below. To understand it we refer to some definitions in his article.

Definition 1. An element $\alpha \in \mathbb{C}_p$ is said to be (p -adically) *non-Liouville* if for $s \in \mathbb{N}$, we have

$$\text{ord}(\alpha + s) = O(\log s) \quad \text{as } s \rightarrow \infty.$$

Definition 2. For each non-Liouville number α the *weight* $w(\alpha)$ is defined by Clark. In this note we only use

$$w(\alpha) = \frac{1}{p-1}$$

for $\alpha \in \mathbb{Z}_p$.

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Definition 3. The *weight* $w(g)$ of a polynomial $g \in \mathbb{C}_p[X]$ whose roots are non-Liouville is the sum of the weights of the roots.

The *ordinal* of a polynomial $\Phi \in \mathbb{C}_p[X]$, say $\sum_{i=0}^n a_i X^i$, is defined by

$$\text{ord } \Phi = \min_i \{\text{ord } a_i\}.$$

(Notice that “minimal” is carelessly mistaken for “maximal” in his article.)

We can summarize Clark’s statement as follows:

Consider a linear differential equation

$$(1) \quad A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \cdots + A_0(x)y = 0$$

where coefficients lie in $\mathbb{C}_p[[x]]$ with nonzero radius of convergence. We may write

$$A_i = x^i \sum_{j=0}^{\infty} a_{ij} x^j \quad (0 \leq i \leq n) \text{ and } a_{i0} \neq 0 \text{ for some } i. \text{ Let } \Phi_j(s) = \sum_{i=0}^n a_{ij} \binom{s}{i} i! \text{ for}$$

$j \geq 0$; then Φ_0 is the indicial polynomial of (1) at $x = 0$.

“Statement”. *If every zero of the indicial polynomial of (1) is non-Liouville, then each power series solution converges for*

$$\text{ord } x > \sup_{j \geq 1} \left\{ \frac{\text{ord } \Phi_0 - \text{ord } \Phi_j}{j} \right\} + w(\Phi_0).$$

2. A NON-LIOUVILLE NUMBER

In order to give the example for which the Statement does not hold, we define a special non-Liouville number in the following lemma.

Lemma. *Let $(e_k)_{k=1}^{\infty}$ be a sequence of positive integers given by $e_1 = 1$ and $e_{k+1} = kp^{e_k}$ ($k \geq 1$), and let $\lambda = \sum_{k=1}^{\infty} p^{e_k}$. Then we have*

- (i) $\varliminf_{n \rightarrow \infty} \sqrt[n]{|n - \lambda|} = 0$;
- (ii) $\lambda \in \mathbb{Z}_p \setminus \mathbb{Z}$;
- (iii) $\text{ord}(\lambda + n) = O(\log n)$ as $n \rightarrow \infty$.

Proof. (i) Let $n_k = \sum_{j=1}^k p^{e_j}$. Then $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers, and we have

$$\sqrt[n_k]{|\lambda - n_k|} = p^{-e_{k+1}/n_k}.$$

Since $p^{e_k} \leq n_k < 2p^{e_k}$, it follows that

$$-\frac{e_{k+1}}{n_k} < -\frac{e_{k+1}}{2p^{e_k}} = -\frac{k}{2},$$

and hence

$$\sqrt[n_k]{|\lambda - n_k|} < p^{-\frac{k}{2}}.$$

Thus we have

$$\lim_{k \rightarrow \infty} \sqrt[n_k]{|\lambda - n_k|} = 0,$$

and therefore

$$\varliminf_{n \rightarrow \infty} \sqrt[n]{|\lambda - n|} = 0.$$

(ii) Clearly $\lambda \in \mathbb{Z}_p \setminus \mathbb{N}^0$. If $-\lambda \in \mathbb{N}$, that is, there is a positive integer m such that $-\lambda = m$, then we have

$$\sqrt[n]{|n - \lambda|} = |n + m|^{\frac{1}{n}} \geq \left(\frac{1}{n + m} \right)^{\frac{1}{n+m} \frac{n+m}{n}},$$

and therefore

$$\lim_{n \rightarrow \infty} \sqrt[n]{|\lambda - n|} = 1,$$

which is contrary to (i).

(iii) Let $n \geq p$, and let

$$n = a_0 + a_1p + \cdots + a_l p^l$$

be the p -adic expansion of n , where $a_l \neq 0$. Then, as $p^l \leq n < p^{l+1}$, we have

$$0 \leq \text{ord}(\lambda + n) \leq l + 2 \leq \frac{\log n}{\log p} + 2,$$

and therefore

$$0 \leq \frac{\text{ord}(\lambda + n)}{\log n} \leq \frac{3}{\log p}.$$

Hence

$$\text{ord}(\lambda + n) = O(\log n) \quad \text{as } n \rightarrow \infty.$$

□

3. EXAMPLE

Now, let λ be the non-Liouville number defined above. Consider the linear differential equation

$$(2) \quad (1 - x)x^2y'' + \{(1 - \lambda) - (2 - \lambda)x\}xy' + \lambda xy = 0.$$

In this equation, we have

$$\begin{aligned} A_2 &= x^2(1 - x), \\ A_1 &= x\{(1 - \lambda) - (2 - \lambda)x\}, \\ A_0 &= \lambda x, \end{aligned}$$

and so

$$\begin{aligned} a_{20} &= 1, & a_{21} &= -1, \\ a_{10} &= 1 - \lambda, & a_{11} &= -(2 - \lambda), \\ a_{00} &= 0, & a_{01} &= \lambda, \\ a_{ij} &= 0 \quad \text{if } j \geq 2. \end{aligned}$$

As $\Phi_j(s) = a_{0j} + a_{1j}s + a_{2j}s(s - 1)$, we have

$$\begin{aligned} \Phi_0(s) &= (1 - \lambda)s + s(s - 1) = s(s - \lambda), \\ \Phi_1(s) &= \lambda - (2 - \lambda)s - s(s - 1) = -s^2 - (1 - \lambda)s + \lambda, \\ \Phi_j(s) &= 0 \quad \text{if } j \geq 2. \end{aligned}$$

The roots of the indicial equation are 0 and λ , and so the requirement of the Statement is satisfied. Since $\text{ord } \lambda \geq 0$ by Lemma (ii), we have

$$\begin{aligned}\text{ord } \Phi_0 &= \min\{\text{ord } 1, \text{ord } \lambda\} = 0, \\ \text{ord } \Phi_1 &= \min\{\text{ord } 1, \text{ord}(1 - \lambda), \text{ord } \lambda\} = 0, \\ \text{ord } \Phi_j &= \infty \quad \text{if } j \geq 2.\end{aligned}$$

This gives

$$\begin{aligned}\sup_{j \geq 1} \left\{ \frac{\text{ord } \Phi_0 - \text{ord } \Phi_j}{j} \right\} + w(\Phi_0) &= \sup_{j \geq 1} \left\{ \frac{-\text{ord } \Phi_j}{j} \right\} + w(0) + w(\lambda) \\ &= \frac{2}{p-1}.\end{aligned}$$

According to the Statement, the power series solution of (2) must converge for

$$\text{ord } x > \frac{2}{p-1},$$

but the equation (2) has the power series solution

$$y = \sum_{n=0}^{\infty} \frac{x^n}{n - \lambda},$$

whose radius of convergence is zero by virtue of Lemma (i).

4. "COROLLARY"

Clark's original proof of Theorem 3 needs the Corollary on p.266 of his article:

"Corollary". *Let h be a monic polynomial whose roots are non-Liouville and in $\{\alpha \in \mathbb{C}_p : |\alpha| \leq 1\}$, and let t be a fixed positive integer greater than all roots of h in \mathbb{Z} . If $s, s' \in \mathbb{Z}, s > s' > t$, then*

$$\sum_{j=s'}^s \text{ord } h(j) = w(h)(s - s') + O(\log s).$$

Professor Dwork suggested to the author that a similar error appears in this statement. Indeed, if it were true we should have the following claim, which is contrary to our Lemma.

Claim. Let $\alpha \in \mathbb{Z}_p \setminus \mathbb{Z}$ such that $\text{ord}(\alpha + s) = O(\log s)$. Then

$$\underline{\lim}_{s \rightarrow \infty} \sqrt[s]{|s - \alpha|} \geq p^{-\frac{1}{p-1}}.$$

In fact, let $h(X) = X - \alpha$. This is the monic polynomial whose root is only α . Therefore

$$w(h) = w(\alpha) = \frac{1}{p-1}.$$

By the Corollary we have

$$\sum_{j=s'}^s \text{ord}(j - \alpha) = \frac{s - s'}{p-1} + O(\log s),$$

where $s > s' > t$. Thus

$$0 \leq \text{ord}(s - \alpha) \leq \sum_{j=s'}^s \text{ord}(j - \alpha) \leq \frac{s-t}{p-1} + M \log s$$

for some $M > 0$, so that

$$|s - \alpha| \geq p^{-\frac{s-t}{p-1} - M \log s},$$

and therefore

$$\underline{\lim}_{s \rightarrow \infty} \sqrt[s]{|s - \alpha|} \geq p^{-\frac{1}{p-1}}.$$

However, this claim contradicts our Lemma when $\alpha = \lambda$.

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