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# A TWO-PARAMETER "BERGMAN SPACE" INEQUALITY

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Abstract. For $f \in L^1([0,1]\times[0,1])$, define $\lambda_R \equiv \langle f, h_{(R)}\rangle$, where $h_{(R)}(x,y) = h_{(I)}(x) \cdot h_{(J)}(y)$ is a tensor product of one-parameter Haar functions. Let $1 < p \leq q < \infty$ and $q \geq 2$. We prove a sufficient condition, which is close to necessary, on double sequences of weights $\{\mu_R\}_R$ and non-negative $v \in L^1([0,1] \times [0,1])$, which ensures that the inequality

$$\left(\sum_R |\lambda_R|^q \mu_R\right)^{1/q} \leq \left(\int_{[0,1]\times[0,1]} |f|^p \, v \, dx\right)^{1/p}$$

holds for all $f \in L^1([0,1] \times [0,1])$. We extend our result to an inequality concerning two-parameter wavelet families.

If $I \subset [0,1]$ is a dyadic interval, we may set

$$h_{(I)}(x) = \begin{cases} |I|^{-1/2} & \text{if } x \in I_L, \\ -|I|^{-1/2} & \text{if } x \in I_R, \\ 0 & \text{if } x \notin I, \end{cases}$$

where $I_L$ and $I_R$ are respectively the left and right halves of $I$, and $|\cdot|$ denotes Lebesgue measure. The function $h_{(I)}$ is the *Haar function* supported on $I$. It is well known (and easy to prove) that $\{h_{(I)}\}_I$ is an orthonormal system in $L^2([0,1])$. If $f \in L^1[0,1]$, we define the *Haar coefficients* $\lambda_I$ of $f$ by:

$$\begin{aligned} \lambda_I &= \langle f, h_{(I)}\rangle \\ &= \int_0^1 f(x)\, h_{(I)}(x)\, dx. \end{aligned}$$

The numbers $\lambda_I$ are "discrete" analogues of wavelet coefficients. They measure, very crudely, how much of the frequency $\approx |I|^{-1}$ the function $f$ has on the interval $I$.

It is easy to see how a single Haar coefficient is controlled by the size of $f$, and in particular how it is affected by small perturbations in $f$. It is more difficult to see how a weighted average of a (possibly infinite) collection of $\lambda_I$'s is so affected. To better understand this dependence, it is natural to consider weighted norm

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inequalities of the following form:

$$\left(\sum_I |\lambda_I|^q \mu_I\right)^{1/q} \le \left(\int_0^1 |f|^p \, v \, dx\right)^{1/p}, \tag{1}$$

where $\mathcal{M} = \{\mu_I\}_I$ is a sequence of non-negative numbers, and $v$ is a non-negative function in $L^1([0,1])$. (The exponents $p$ and $q$ are assumed to lie between 1 and $\infty$.) One then looks for sufficient conditions on $\{\mu_I\}_I$ and $v$ which imply that (1) holds for all $f \in L^1([0,1])$. Naturally, one hopes that such sufficient conditions will be close to necessary.

Richard Wheeden and the author investigated this question in [WW]. They found both sufficient and necessary conditions for (1), that used an auxiliary function defined from $v$. Let $p'$ be the conjugate exponent to $p$. They defined $\sigma \equiv v^{1-p'}$. We claim that, if (1) is true, then

$$\frac{\left(\int_I \sigma \, dx\right)^{1/p'} \mu_I^{1/q}}{|I|^{1/2}} \le 1 \tag{2}$$

must hold for all dyadic intervals $I$. (In particular, this implies that $\int_I \sigma$ is finite whenever $\mu_I \neq 0$.) To obtain (2), set, for $n = 1, 2, \ldots$, $v_n = \max(v, 1/n)$ and $\sigma_n = v_n^{1-p'}$. Clearly, $\sigma_n \in L^\infty$, and $\int_I \sigma_n > 0$ for all $I$ (because $v$ is finite a.e.). If (1) holds for $v$, then it holds for $v_n \ge v$. Fix a dyadic interval $I$ and set $f_n(x) = \sigma_n \cdot (\chi_{I_L}(x) - \chi_{I_R}(x))$. If we plug $f_n$ into (1) (with $v_n$ in place of $v$) and use the fact that $\sigma_n^p v_n = \sigma_n$, we obtain

$$\frac{\left(\int_I \sigma_n \, dx\right)^{1/p'} \mu_I^{1/q}}{|I|^{1/2}} \le 1.$$

Inequality (2) follows now by monotone convergence.

The sufficient condition they proved (valid, however, only for a limited range of $p$'s and $q$'s) is just a bit more restrictive. For any non-negative real number $\eta$, and any dyadic interval $I$, they defined

$$\sigma(I,\eta) = \int_I \sigma(x) \, \log^\eta(e + \sigma(x)/\sigma_I) \, dx,$$

where $\sigma_I$ denotes $\sigma$'s average over $I$. They proved ([WW], Theorem 1):

**Theorem 1.** *Let $v$, $\sigma$, and $\mathcal{M}$ be as above. Let $1 < p \le q < \infty$ and $q \ge 2$. Suppose that $\eta > p'/2$. There is a positive constant $C = C(p,q,\eta)$ such that if*

$$\frac{\sigma(I,\eta)^{1/p'} \mu_I^{1/q}}{|I|^{1/q}} \le C$$

*for all dyadic intervals $I$, then (1) holds.*

The authors also proved an analogous result for wavelet expansions ([WW], Theorem 2).

Inequality (1) has a natural reformulation in a two-parameter setting. If $R = I \times J$ is a Cartesian product of dyadic intervals, we can set $h_{(R)}(x,y) = h_{(I)}(x) \cdot h_{(J)}(y)$, the tensor product of two "one-parameter" Haar functions, and define $\lambda_R \equiv \langle f, h_{(R)} \rangle$ for $f \in L^1([0,1] \times [0,1])$. The coefficients $\lambda_R$ have a meaning analogous to that of the $\lambda_I$'s, except that now the "frequencies" are two-dimensional vectors and are supported on rectangles. We want strong (i.e., close to necessary) conditions on double sequences of non-negative weights $\mathcal{N} = \{\mu_R\}_R$ and non-negative functions $v \in L^1([0,1] \times [0,1])$ which ensure that

$$\left(\sum_R |\lambda_R|^q \mu_R\right)^{1/q} \leq \left(\int_{[0,1]\times[0,1]} |f|^p \, v \, dx\right)^{1/p} \tag{3}$$

holds for all $f \in L^1([0,1] \times [0,1])$. (As before, we assume that $p$ and $q$ lie strictly between 1 and $\infty$.)

Set $\sigma = v^{1-p'}$. An argument like the one given above shows that, in order for (3) to hold, we must have

$$\frac{\left(\int_R \sigma \, dx \, dy\right)^{1/p'} \mu_R^{1/q}}{|R|^{1/2}} \leq 1$$

for all double-dyadic rectangles $R$. This fact lends superficial plausibility to the idea that a sufficient condition for (3) should have a form like that given in Theorem 1, at least for the same range of $p$'s and $q$'s.

For $R$ a double-dyadic rectangle, $\sigma$ a non-negative measurable function, and $\eta$ a non-negative number, define

$$\tilde{\sigma}(R, \eta) \equiv \int_R \sigma(x, y) \, \log^\eta(e + \sigma/\sigma_R) \, dx \, dy,$$

where $\sigma_R$ is $\sigma$'s average over $R$. (We are calling this new functional '$\tilde{\sigma}(R, \eta)$', instead of '$\sigma(R, \eta)$', because we still have need of the latter.) Now we can state our theorem.

**Theorem 2.** *Let $v$ be a non-negative function in $L^1([0,1] \times [0,1])$. Let $\mathcal{N} = \{\mu_R\}_R$ be a sequence of non-negative numbers indexed over the double-dyadic rectangles in $[0,1] \times [0,1]$. Let $1 < p \leq q < \infty$ and $q \geq 2$. Set $\sigma = v^{1-p'}$. For every $\eta > p'$, there is a positive constant $C = C(p, q, \eta)$ such that if*

$$\frac{\tilde{\sigma}(R, \eta)^{1/p'} \mu_R^{1/q}}{|R|^{1/2}} \leq C \tag{4}$$

*for all $R$, then (3) holds for all $f \in L^1([0,1] \times [0,1])$.*

*Proof.* Up to a point, the proof follows that of Theorem 1 in [WW]. Let $\mathcal{R}$ denote the family of all double-dyadic rectangles in $[0,1] \times [0,1]$. For $g \colon \mathcal{R} \mapsto \mathbf{R}$ a *finitely supported* function, set

$$Tg(x, y) = \sum_R g(R) \mu_R h_{(R)}(x, y).$$

The sum is well-defined because $g$ has finite support. By duality, inequality (3) will hold if

$$\left(\int_{[0,1]\times[0,1]} |Tg|^{p'} \sigma \, dx \, dy\right)^{1/p'} \leq \left(\sum_R |g(R)|^{q'} \mu_R\right)^{1/q'} \tag{5}$$

for all such $g$.

As in the proof of Theorem 1 in [WW], the proof that (4) implies (5) breaks into two cases: $p' < 2$ and and $p' \geq 2$. The first case follows from a fairly straightforward iteration of one-dimensional results, and we will treat it first.

Fix $y \in [0,1]$. Set $R_y(x) = Tg(x,y)$ and $\nu_y(x) = \sigma(x,y)$. We may write $R_y(x)$ as a linear combination of one-dimensional Haar functions:

$$\begin{aligned} R_y(x) &= \sum_I \left( \sum_{J \ni y} g(I \times J) \mu_{I \times J} h_{(J)}(y) \right) \cdot h_{(I)}(x) \\ &= \sum_I \gamma_I^{(y)} h_{(I)}(x). \end{aligned}$$

Let $\tau = \eta/2 > p'/2$. Following the argument in [WW] (Theorem 1), we see that

$$\int_0^1 |R_y(x)|^{p'} \, \nu_y(x) \, dx \le C \sum_I \frac{|\gamma_I^{(y)}|^{p'}}{|I|^{p'/2}} \nu_y(I, \tau),$$

where the constant $C$ depends on $p'$ and $\tau$.

By Young's Inequality, we can dominate $\nu_y(I,\tau)$ by an integral of the form

$$\int_I \nu_y(x) \, \psi(I,x,y) \, dx, \tag{6}$$

where $\psi(I,x,y)$ satisfies

$$\frac{1}{|I|} \int_I \exp(\psi(I,x,y)^{1/\tau}) \, dx \le 6. \tag{7}$$

(See [W], Theorem 2.1.) Clearly, the functions $\psi(I,x,y)$ can be taken to be measurable in $(x,y)$.

Call the quantity in (6) $\Omega^I(y)$. We have shown that, for every $y$,

$$\int_0^1 |R_y(x)|^{p'} \, \nu_y(x) \, dx \le C(\tau, p') \sum_I \frac{|\gamma_I^{(y)}|^{p'}}{|I|^{p'/2}} \Omega^I(y).$$

Let us now take *one term* from the preceding right-hand sum and integrate it in $y$. Recalling that $\gamma_I^{(y)}$ is itself a sum of Haar functions (in $y$!), a verbatim repetition of the argument from [WW] shows that

$$\begin{aligned} \int_0^1 \frac{|\gamma_I^{(y)}|^{p'}}{|I|^{p'/2}} \Omega^I(y) \, dy &\le \frac{C(\tau,p')}{|I|^{p'/2}} \sum_J \frac{|g(I \times J)|^{p'} \mu_R^{p'}}{|J|^{p'/2}} \Omega^I(J, \tau) \\ &\le \frac{C'(\tau,p')}{|I|^{p'/2}} \sum_J \frac{|g(I \times J)|^{p'} \mu_R^{p'}}{|J|^{p'/2}} \\ &\qquad \times \int_{I \times J} \sigma(x,y) \, \psi(I,x,y) \, \phi(I,J,y) \, dx \, dy, \end{aligned} \tag{8}$$

where each function $\phi(I,J,y)$ satisfies

$$\frac{1}{|J|} \int_J \exp(\phi(I,J,y)^{1/\tau}) \, dy \le 6. \tag{9}$$

Plugging (8) back into our sum yields:

$$\begin{aligned} &\int_{[0,1]\times[0,1]} |Tg|^{p'} \, \sigma \, dx \, dy \\ &\qquad \le C''(\tau,p') \sum_{R = I \times J} \frac{|g(I \times J)|^{p'} \mu_R^{p'}}{|R|^{p'/2}} \int_{I \times J} \sigma(x,y) \, \psi(I,x,y) \, \phi(I,J,y) \, dx \, dy. \end{aligned}$$

Since $\psi(I,x,y)$ and $\phi(I,J,y)$ satisfy respectively (7) and (9) uniformly on the (respectively) $x$- and $y$-slices of $R = I \times J$, the Cauchy-Schwarz inequality implies that their product, $\rho(I,J,x,y)$, must satisfy

$$\frac{1}{|R|}\int_I \exp(\rho(I,J,x,y)^{1/2\tau})\,dx\,dy \le 6.$$

Now Young's Inequality (again) implies that

$$\int_{I\times J} \sigma(x,y)\,\psi(I,x,y)\,\phi(I,J,y)\,dx\,dy \le C\tilde{\sigma}(I\times J, 2\tau).$$

Thus,

$$\begin{aligned}\int_{[0,1]\times[0,1]} |Tg|^{p'}\,\sigma\,dx\,dy &\le C''(\tau,p')\sum_{R=I\times J}\frac{|g(I\times J)|^{p'}\mu_R^{p'}}{|R|^{p'/2}}\tilde{\sigma}(R,2\tau)\\ &= C''(\tau,p')\sum_{R}\frac{|g(R)|^{p'}\mu_R^{p'}}{|R|^{p'/2}}\tilde{\sigma}(R,\eta),\end{aligned}$$

since $\eta = 2\tau$. The hypothesis of Theorem 2 implies that the last line is bounded by

$$C\sum_R |g(R)|^{p'}\mu_R^{p'/q'} \le C\left(\sum_R |g(R)|^{q'}\mu_R\right)^{p'/q'},$$

where the right-hand side follows because $p' \ge q'$. This proves the first case.

The proof of the second ($p' \ge 2$) case which we will give does *not* follow from a simple iteration of the corresponding one-dimensional argument. The one-parameter proof (Theorem 1 from [WW]) used some fairly recent results about one-parameter weighted Littlewood-Paley theory. Unfortunately, we do not yet have proofs of the analogous two-parameter Littlewood-Paley results, except in the case where we are mapping $L^2 \mapsto L^2$. Nor do we prove such inequalities here. Instead, we prove the “Bergman space” inequality (for $p' \ge 2$) using *only* a weighted $L^2 \mapsto L^2$ result. Our argument also works in the one-parameter setting, and in that case it yields a much simpler proof than the one given in [WW].

We will need the following result from [W] (Theorem 2.1): *Let $\mathcal{F}$ be a finite family of double-dyadic rectangles and let $\tau > 2$. There is a constant $C_\tau$ such that, for all functions*

$$f(x,y) = \sum_{R\in\mathcal{F}} \lambda_R h_{(R)}(x,y)$$

*and all non-negative $\nu \in L^1([0,1]\times[0,1])$,*

$$\int_{[0,1]\times[0,1]} |f|^2\,\nu\,dx\,dy \le C_\tau \sum_{R\in\mathcal{F}} \frac{|\lambda_R|^2}{|R|}\tilde{\nu}(R,\tau). \tag{10}$$

Let us now take $g\colon \mathcal{R} \mapsto \mathbf{R}$ to be a finitely supported function. We wish to prove (5). Let $\rho$ be the dual exponent to $p'/2$ (which, recall, is $\ge 1$). There exists a non-negative $\phi \in L^\rho(\sigma)$ with $\|\phi\|_{L^\rho(\sigma)} = 1$ such that

$$\int_{[0,1]\times[0,1]} |Tg|^{p'}\,\sigma\,dx\,dy = \left(\int_{[0,1]\times[0,1]} |Tg|^2\,\phi\,\sigma\,dx\,dy\right)^{p'/2}. \tag{11}$$

We set $\aleph = \phi \cdot \sigma$, and use (10) to bound the right-hand side of (11) by

$$\left( C_\tau \sum_R \frac{|g(R)\mu_R|^2}{|R|} \tilde{\aleph}(R, \tau) \right)^{p'/2},$$

for $\tau = 2\eta/p' > 2$.

By Young's Inequality, the quantity $\tilde{\aleph}(R, \tau)$ can be dominated by

$$C \int_R \aleph(x, y)\, \psi(R, x, y)\, dx\, dy = C \int_R \phi(x, y) \psi(R, x, y)\, \sigma\, dx\, dy, \tag{12}$$

where $\psi(R, x, y)$ satisfies

$$\frac{1}{|R|} \int_R \exp(\psi(R, x, y)^{1/\tau})\, dx\, dy \leq 6.$$

Because of Hölder's Inequality (and the fact that $\|\phi\|_{L^p(\sigma)} \leq 1$), the right-hand side of (12) is less than or equal to

$$C \left( \int_R \psi(R, x, y)^{p'/2}\, \sigma\, dx\, dy \right)^{2/p'}. \tag{13}$$

The function $\beta(x, y) = \psi(R, x, y)^{p'/2}$ satisfies

$$\frac{1}{|R|} \int_R \exp(\beta(x, y)^{2/(p'\tau)})\, dx\, dy \leq 6.$$

Therefore, Young's Inequality implies that (13) is less than or equal to $C\tilde{\sigma}(R, p'\tau/2)^{2/p'}$. When we substitute this back into (11), we get:

$$\int_{[0,1]\times[0,1]} |Tg|^{p'}\, \sigma\, dx\, dy \leq C_\tau \left( \sum_R \frac{|g(R)\mu_R|^2}{|R|} \tilde{\sigma}(R, p'\tau/2)^{2/p'} \right)^{p'/2}. \tag{14}$$

Since $\eta = p'\tau/2$, the hypothesis of Theorem 2 implies that the right-hand side of (14) is dominated by

$$C'_\eta \left( \sum_R |g(R)|^2 \mu_R^{2/q'} \right)^{p'/2},$$

which in turn is less than or equal to

$$C'_\eta \left( \sum_R |g(R)|^{q'} \mu_R \right)^{p'/q'},$$

because $q' \leq 2$. Theorem 2 is proved.

Theorem 2 has an immediate corollary relating to wavelet expansions. Let $\psi$ be a smooth function supported in $[-1, 2]$, and which also satisfies $\int_{\mathbf{R}} \psi = 0$. If $I \subset \mathbf{R}$ is a dyadic interval with left endpoint $x^*$, we set $\psi_{(I)}(x) = |I|^{-1/2}\psi((x - x^*)/\ell(I))$, where $\ell(I)$ denotes the length of $I$. (If $\psi$ is chosen cleverly, the family $\mathcal{G} = \{\psi(I)\}_I$, called a *wavelet system*, is an orthonormal basis for $L^2(\mathbf{R})$.)

We wish to consider a two-parameter version of $\mathcal{G}$. For $R = I \times J \subset \mathbf{R}^2$, a double-dyadic rectangle, set

$$\Psi_{(R)}(x, y) = \psi_{(I)}(x) \cdot \psi_{(J)}(y).$$

Set $\mathcal{DG} = \{\Psi_{(R)}\}_R$. Just as with $\mathcal{G}$, the right choice of $\psi$ makes $\mathcal{DG}$ into an orthonormal basis—this time for $L^2(\mathbf{R}^2)$.

For $f \in L^1_{loc}(\mathbf{R}^2)$, set $\lambda^*_R = \langle f, \Psi_{(R)}\rangle$. This coefficient has the same significance as $\lambda_R$, but it is, as a rule, a more reliable and useful measure of $f$'s local spectrum than $\lambda_R$.

For the reasons given in the introduction, it is natural to ask for what non-negative weights $v \in L^1_{loc}(\mathbf{R}^2)$ and double sequences of weights $\{\mu_R\}$ the inequality

$$\left(\sum_R |\lambda^*_R|^q \mu_R\right)^{1/q} \le \left(\int_R |f|^p\, v\, dx\right)^{1/p} \tag{15}$$

holds for all $f \in L^1_{loc}(\mathbf{R}^2)$. Fortunately, one answer (along with its proof) is virtually the same as that for the double-dyadic Haar coefficients, at least if we consider the same range of $p$'s and $q$'s. For $I$ a dyadic interval, let $\tilde{I}$ denote $I$'s triple. If $R = I \times J$, we set $\tilde{R} = \tilde{I} \times \tilde{J}$. In [W] (Theorem 2.2) it is proved that if $\mathcal{DF} \subset \mathcal{DG}$ is any finite family and

$$g(x,y) = \sum_{R \in \mathcal{DF}} \gamma_R \Psi_{(R)}(x,y),$$

where the $\gamma_R$'s are real numbers, then, for any $\tau > 2$ and any weight $\sigma \in L^1_{loc}(\mathbf{R}^2)$,

$$\int_{\mathbf{R}^2} |g|^2\, \sigma\, dx \le C_{\tau,\psi} \sum_R \frac{|\gamma_R|^2}{|R|} \tilde{\sigma}(\tilde{R}, \tau). \tag{16}$$

This fact is the key to the “hard” ($p' \ge 2$) case of the following

**Theorem 3.** *Let $\mathcal{DG}$ be a two-parameter wavelet system, given as above. Let $v$ be a non-negative function in $L^1_{loc}(\mathbf{R}^2)$. Let $\mathcal{N} = \{\mu_R\}_R$ be a sequence of non-negative numbers indexed over the double-dyadic rectangles in $\mathbf{R}^2$. Let $1 < p \le q < \infty$ and $q \ge 2$. Set $\sigma = v^{1-p'}$. For every $\eta > p'$, there is a positive constant $C = C(p,q,\eta,\psi)$ such that if*

$$\frac{\tilde{\sigma}(\tilde{R}, \eta)^{1/p'} \mu_R^{1/q}}{|R|^{1/2}} \le C \tag{17}$$

*for all $R$, then (15) holds for all $f \in L^1_{loc}(\mathbf{R}^2)$.*

*Proof.* Let $g \colon \mathcal{DG} \mapsto \mathbf{R}$ have finite support and define

$$\tilde{T}g(x,y) = \sum_{R \in \mathcal{DG}} g(R)\mu_R \Psi_{(R)}(x,y),$$

analogous to $Tg$ defined earlier. We wish to establish the inequality

$$\left(\int_{\mathbf{R}^2} |\tilde{T}g|^{p'}\, \sigma\, dx\, dy\right)^{1/p'} \le \left(\sum_R |g(R)|^{q'} \mu_R\right)^{1/q'}$$

under the assumption that (17) holds.

As before, the argument breaks into two cases; and, just as before, the '$p' \leq 2$' case follows from a direct iteration of one-parameter results. Note that, for fixed $y$,

$$\begin{aligned}\tilde{T}g(x,y) &= \sum_I \left( \sum_{y \in \tilde{J}} g(\tilde{I} \times \tilde{J}) \mu_R \psi_{(J)}(y) \right) \cdot \psi_{(I)}(x) \\ &= \sum_I \gamma_I^y \psi_{(I)}(x).\end{aligned}$$

Let $\tau = \eta/2 > p'/2$. By the proof of Theorem 2 (Subcase $1'$: $2 < p < \infty$) in [WW], we see that, for every $y$,

$$\int_{\mathbf{R}} |\tilde{T}g|^{p'} \sigma(x,y)\, dx \leq C_{\tau,\psi} \sum_I \frac{|\gamma_I^y|^{p'}}{|I|^{p'/2}} \nu_y(\tilde{I}, \tau),$$

where, as before, we have set $\nu_y(x) = \sigma(x,y)$. But this is precisely the same sort of expression we got in the earlier Haar function case! If we integrate in $y$ now, and combine our Young's Inequality argument with the result from Theorem 2 in [WW] (which is the "smooth" analogue of Theorem 1), we get the '$p' \leq 2$' result here as well.

The other case is just as easy. Let $\rho$ be the dual exponent to $p'/2 \geq 1$. There is a non-negative $\phi \in L^\rho(\sigma)$ such that $\|\phi\|_{L^\rho(\sigma)} = 1$ and

$$\int_{\mathbf{R}^2} |\tilde{T}g|^{p'} \sigma\, dx\, dy = \left( \int_{\mathbf{R}^2} |\tilde{T}g|^2 \phi\, \sigma\, dx\, dy \right)^{p'/2}.$$

By inequality (16) above, the right-hand integral is dominated by

$$C_{\eta,\psi} \sum_R \frac{|g(R)\mu_R|^2}{|R|} \tilde{\aleph}(\tilde{R}, \eta),$$

where we set $\aleph = \sigma \cdot \phi$. Once again, this is exactly what we got in the Haar function case, and the proof concludes the same way. QED

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