

INTERSECTION OF SETS WITH n -CONNECTED UNIONS

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ABSTRACT. We show that if n sets in a topological space are given so that all the sets are closed or all are open, and for each $k \leq n$ every k of the sets have a $(k - 2)$ -connected union, then the n sets have a point in common. As a consequence, we obtain the following starshaped version of Helly's theorem: If every $n + 1$ or fewer members of a finite family of closed sets in \mathbb{R}^n have a starshaped union, then all the members of the family have a point in common. The proof relies on a topological KKM-type intersection theorem.

1. INTRODUCTION

The purpose of this paper is to generalize to an arbitrary finite family of sets the following elementary fact: *If in a topological space two nonempty sets, both closed or both open, have a pathwise connected union, then they have a point in common.* To this end, we use the notion of n -connectedness which is a natural generalization of pathwise connectedness. Let us recall the definition.

For any integer $n \geq -1$, we denote by Δ_{n+1} the unit $(n + 1)$ -simplex and by $\partial\Delta_{n+1}$ its boundary. A topological space C is said to be n -connected if every continuous map $f : \partial\Delta_{n+1} \rightarrow C$ has a continuous extension $g : \Delta_{n+1} \rightarrow C$.

Clearly, (-1) -connected means nonempty and 0 -connected means pathwise connected. It is also easily seen that for a pathwise connected space, 1 -connected is equivalent to simply connected. Convex sets and starshaped sets in topological vector spaces, and, more generally, contractible spaces are n -connected for every n .

Our main result (Theorem 3) reads as follows: *If n sets in a topological space are given so that all the sets are closed or all are open and for each $k \leq n$ every k of the sets have a $(k - 2)$ -connected union, then the n sets have a point in common.* For $n = 2$ we exactly recover the above-mentioned fact. This extension is equivalent to Brouwer's fixed point theorem.

Applications to Helly-type theorems are considered. In particular, the following generalization of results of Breen [2, 3] is established (Theorem 5): *If every $n + 1$ or fewer members of a finite family of closed sets in \mathbb{R}^n have a starshaped union, then all the members of the family have a point in common.*

The basic tool for our study is an intersection theorem (Theorem 2) obtained by combining the well-known theorem of Knaster-Kuratowski-Mazurkiewicz with a selection-extension property for families of n -connected sets (Theorem 1). No

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notion from algebraic topology is used. The ideas of this approach come from Horvath [11] and Lassonde [14].

For a comprehensive account on Helly-type results, the reader is referred to the survey papers of Danzer, Grünbaum, Klee [4] and Eckhoff [7].

2. SELECTION-EXTENSION PROPERTY

By *polytope* we understand a simplicial CW-complex in the sense of Whitehead. We recall the definitions. A *triangulation* is a collection \mathcal{T} of geometric closed simplexes such that every face of a simplex in \mathcal{T} is itself a simplex in \mathcal{T} , and the intersection of any two simplexes in \mathcal{T} is a face of both of them. We denote by $\mathcal{T}^{(n)}$ the collection of all simplexes of \mathcal{T} of dimension less than or equal to n .

A *polytope* is a topological space P together with a triangulation \mathcal{T} such that $P = \bigcup\{\sigma \mid \sigma \in \mathcal{T}\}$, and P is supplied with the CW-topology, i.e. $U \subseteq P$ is open if and only if for each $\sigma \in \mathcal{T}$, $U \cap \sigma$ is open in the Euclidean topology of σ . We denote by $P^{(n)}$ the n -skeleton of P , i.e. the subpolytope of P with triangulation $\mathcal{T}^{(n)}$. A *subpolytope* of P is a polytope $Q \subseteq P$ with a triangulation $\mathcal{S} \subseteq \mathcal{T}$.

The following theorem, though more general than required for the applications considered in this paper, is worth stating explicitly for future use. It strengthens a result of Horvath [11, Theorem 1] in several aspects; the proof is similar.

Theorem 1. *Let X be a topological space, P a polytope with triangulation \mathcal{T} , $Q \subseteq P$ a subpolytope with triangulation $\mathcal{S} \subseteq \mathcal{T}$, and $\{C_\sigma \mid \sigma \in \mathcal{T}\}$ a family of subsets of X such that:*

- (a) $C_\sigma \subseteq C_\tau$ for every $\sigma, \tau \in \mathcal{T}$ with $\sigma \subseteq \tau$,
- (b) C_σ is $(\dim \sigma - 1)$ -connected for every $\sigma \in \mathcal{T} \setminus \mathcal{S}$.

Then any continuous map $f : Q \rightarrow X$ satisfying $f(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{S}$ can be extended to a continuous map $g : P \rightarrow X$ satisfying $g(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{T}$.

Proof. Since for any vertex $x \in \mathcal{T}^{(0)}$ the set $C_{\{x\}}$ is not empty because of (b), the map $f_0 : Q \cup P^{(0)} \rightarrow X$ given by $f_0(x) = f(x)$ for $x \in Q$, and $f_0(x) =$ any point in $C_{\{x\}}$ for $x \in P^{(0)} \setminus Q$ is a continuous extension of f such that $f_0(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{S} \cup \mathcal{T}^{(0)}$. Starting from f_0 , we define inductively a sequence of continuous extensions $f_n : Q \cup P^{(n)} \rightarrow X$ satisfying $f_n(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n)}$. The result then follows by taking $g : P \rightarrow X$ such that $g(x) = f_n(x)$ whenever $x \in Q \cup P^{(n)}$.

Suppose f_n is constructed. The required extension of f_n is obtained by pasting together a family of maps $f_{n+1}^\sigma : \sigma \rightarrow C_\sigma$, $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n+1)}$, defined as follows. For $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n)}$, simply let $f_{n+1}^\sigma = f_n|_\sigma$. Otherwise, write $\partial\sigma = \sigma_1 \cup \dots \cup \sigma_k$, where each σ_i belongs to $\mathcal{T}^{(n)}$; for each $i = 1, \dots, k$, we have $f_n(\sigma_i) \subseteq C_{\sigma_i}$ by construction and $C_{\sigma_i} \subseteq C_\sigma$ by (a), so that $f_n(\partial\sigma) \subseteq C_\sigma$: let $f_{n+1}^\sigma : \sigma \rightarrow C_\sigma$ be a continuous extension of $f_n|_{\partial\sigma} : \partial\sigma \rightarrow C_\sigma$ (which exists because of (b)). Since for any different members σ, τ in $\mathcal{S} \cup \mathcal{T}^{(n+1)}$ we have $f_{n+1}^\sigma|_{\sigma \cap \tau} = f_n|_{\sigma \cap \tau} = f_{n+1}^\tau|_{\sigma \cap \tau}$, there is a map $f_{n+1} : Q \cup P^{(n+1)} \rightarrow X$ which is an extension of each f_{n+1}^σ . Obviously, such a map is continuous for the CW-topology, extends f_n , and satisfies $f_{n+1}(\sigma) \subseteq C_\sigma$ for all $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n+1)}$. \square

As an immediate special case of Theorem 1, we mention the following well-known property (see for example Eilenberg [8, p. 241]):

Corollary. *A topological space X is k -connected for every $k \leq n$ if and only if for any polytope P and any subpolytope $Q \subseteq P$, any continuous $f : Q \rightarrow X$ extends continuously over $Q \cup P^{(n+1)}$.*

Proof. The “if part” is obvious, and the “only if part” follows from Theorem 1 applied to the polytope $Q \cup P^{(n+1)}$ and the sets C_σ given by $C_\sigma = X$ for every $\sigma \in \mathcal{S} \cup \mathcal{T}^{(n+1)}$. □

3. KKM-TYPE INTERSECTION THEOREMS

We denote by $\mathcal{F}(\Delta_n)$ the set of all faces of Δ_n , and for any $\sigma \in \mathcal{F}(\Delta_n)$ we denote by $\sigma^{(0)}$ the set of all vertices of σ . The Knaster-Kuratowski-Mazurkiewicz theorem can be formulated as follows: *If a family $\{A_i \mid i \in \Delta_n^{(0)}\}$ of subsets of Δ_n is such that all the sets are closed or all are open, and each face σ of Δ_n is contained in the corresponding union $\bigcup\{A_i \mid i \in \sigma^{(0)}\}$, then there is a point common to all the sets.* (The original statement deals only with closed sets; a proof that open sets may be used equivalently is given in Lassonde [14].)

By combining Theorem 1 with the Knaster-Kuratowski-Mazurkiewicz theorem we get our basic topological intersection theorem:

Theorem 2. *In a topological space, let $\{A_i \mid i \in \Delta_n^{(0)}\}$ be a family of sets, all closed or all open, and let $\{C_\sigma \mid \sigma \in \mathcal{F}(\Delta_n)\}$ be an associated family of sets such that:*

- (a) $C_\sigma \subseteq C_\tau$ for every $\sigma, \tau \in \mathcal{F}(\Delta_n)$ with $\sigma \subseteq \tau$,
- (b) C_σ is $(\dim \sigma - 1)$ -connected for every $\sigma \in \mathcal{F}(\Delta_n)$.

Then the following equivalent assertions hold:

- (1) *If each C_σ is contained in the corresponding union $\bigcup\{A_i \mid i \in \sigma^{(0)}\}$, then C_{Δ_n} contains a point of the intersection $\bigcap\{A_i \mid i \in \Delta_n^{(0)}\}$;*
- (2) *If C_{Δ_n} is contained in the union $\bigcup\{A_i \mid i \in \Delta_n^{(0)}\}$, then some C_σ contains a point of the corresponding intersection $\bigcap\{A_i \mid i \in \sigma^{(0)}\}$.*

Proof. Denote by X the underlying topological space, and assume that the condition of assertion (1) is satisfied. By Theorem 1, a continuous $g : \Delta_n \rightarrow X$ exists such that for each face σ of Δ_n we have $g(\sigma) \subseteq C_\sigma$, which in view of the assumption implies $g(\sigma) \subseteq \bigcup\{A_i \cap C_{\Delta_n} \mid i \in \sigma^{(0)}\}$. Thus, each face σ of Δ_n is contained in the corresponding union $\bigcup\{g^{-1}(A_i \cap C_{\Delta_n}) \mid i \in \sigma^{(0)}\}$. We derive from the theorem of Knaster-Kuratowski-Mazurkiewicz that the family $\{g^{-1}(A_i \cap C_{\Delta_n}) \mid i \in \Delta_n^{(0)}\}$ has a nonempty intersection. Consequently, the intersection $\bigcap\{A_i \cap C_{\Delta_n} \mid i \in \Delta_n^{(0)}\}$ is not empty, proving assertion (1).

As for assertion (2), it suffices to observe that it is the contraposition form of assertion (1) applied to the family $\{X \setminus A_i \mid i \in \Delta_n^{(0)}\}$. □

Ky Fan’s extension of Knaster-Kuratowski-Mazurkiewicz’s theorem [9] and Ky Fan’s matching theorem [10] are obtained respectively from assertion (1) and assertion (2) of the following:

Corollary. *In a convex subset of a vector space supplied with the finite topology, let $\{A_i \mid i \in I\}$ be a finite family of sets, all closed or all open, and let $\{x_i \mid i \in I\}$ be a family of points, indexed by the same set I . Then the following (equivalent) assertions hold:*

(1) If for every nonempty subset $J \subseteq I$ the convex hull of $\{x_j \mid j \in J\}$ is contained in the corresponding union $\bigcup\{A_j \mid j \in J\}$, then the convex hull of $\{x_i \mid i \in I\}$ contains a point of the intersection $\bigcap\{A_i \mid i \in I\}$.

(2) If the convex hull of $\{x_i \mid i \in I\}$ is contained in the union $\bigcup\{A_i \mid i \in I\}$, then for some nonempty subset $J \subseteq I$ the convex hull of $\{x_j \mid j \in J\}$ contains a point of the corresponding intersection $\bigcap\{A_j \mid j \in J\}$.

Proof. We may assume that the set of indices I is the set of vertices of Δ_n . The result then follows from Theorem 2 with C_σ being, for every $\sigma \in \mathcal{F}(\Delta_n)$, the convex hull of $\{x_j \mid j \in \sigma^{(0)}\}$ (such sets are evidently contractible when supplied with the Euclidean topology). \square

4. KLEE-TYPE INTERSECTION THEOREMS

Our main result below provides topological generalizations of the familiar theorem of Klee [12]: *If n closed convex sets in a topological vector space are such that their union is convex and the intersection of every $n - 1$ of them is nonempty, then all the sets have a point in common.*

Theorem 3. *Let n sets in a topological space be given so that all the sets are closed or all are open, and either of the following properties is satisfied:*

(1) *For each $k \leq n$ the union of every k of the sets is $(k - 2)$ -connected; or*

(2) *The union of the n sets is $(n - 2)$ -connected and for each $k \leq n - 1$ the intersection of every k of the sets is $(n - k - 2)$ -connected.*

Then all the sets have a point in common.

Proof. Let $\{A_i \mid i \in \Delta_{n-1}^{(0)}\}$ be a family of n sets satisfying the conditions of the theorem. We define an associated family of sets $\{C_\sigma \mid \sigma \in \mathcal{F}(\Delta_{n-1})\}$ satisfying the requirements of Theorem 2 as follows. In case (1), we let $C_\sigma = \bigcup\{A_i \mid i \in \sigma^{(0)}\}$ so that assertion (1) of Theorem 2 applies. In case (2), we let $C_\sigma = \bigcup\{A_i \mid i \in \sigma^{(0)}\}$ for $\sigma = \Delta_{n-1}$ and $C_\sigma = \bigcap\{A_i \mid i \notin \sigma^{(0)}\}$ for $\sigma \neq \Delta_{n-1}$, so that assertion (2) of Theorem 2 gives the result. \square

The following immediate consequence of Theorem 3 is worth mentioning:

Corollary. *A family of n closed convex sets in a topological vector space has a nonempty intersection if (and only if) the union of the n sets is $(n - 2)$ -connected and the intersection of every $n - 1$ of them is nonempty.*

This result, which is a slight modification of Klee's theorem, implies the theorem of Brouwer stating that *the n -sphere S^n is not n -connected*. Indeed, consider the n -dimensional faces of Δ_{n+1} : they form a family of $n + 2$ closed convex sets in \mathbb{R}^{n+2} , every intersection of $n + 1$ of them is nonempty, but the whole intersection is empty. Hence their union, which is $\partial\Delta_{n+1}$, is not n -connected, so S^n is not n -connected either. Since this statement is equivalent to Brouwer's fixed point theorem, which is itself equivalent to the theorem of Knaster-Kuratowski-Mazurkiewicz, we infer that the above corollary is equivalent to all these results.

Remark. Case (2) of Theorem 3 is closely related to results of Dugundji for metric spaces (see [5, Theorem 4.1] for the open version, and [6, Theorem 5.2] for the closed version). Dugundji's assumptions are: *The singular $(n - 2)$ -homology group of the union of the n sets is trivial or a torsion group, and for each $k \leq n - 1$ the intersection of every k of the sets is p -connected for every $p \leq n - k - 2$.* In case (2)

of Theorem 3 the hypothesis on the union is stronger but the hypotheses on the intersections are weaker.

5. HELLY-TYPE INTERSECTION THEOREMS

We now turn to discussing Helly's theorem which asserts that *a finite family of convex sets in \mathbb{R}^n has a nonempty intersection if (and only if) the intersection of every $n + 1$ members of the family is nonempty*. As is well known, Helly's theorem can easily be proved by combining Klee's theorem with Carathéodory's theorem (see for example Berge [1, p. 173]).

Recall that a subset C of a vector space is said to be *starshaped* if there is a point $c \in C$ such that for every $x \in C$ the segment $[c, x]$ lies in C . Clearly, if the intersection of convex sets is nonempty, then their union is starshaped. Hence, the following result of Breen [2] is a generalization of Helly's theorem for closed sets: *A finite family of closed convex sets in \mathbb{R}^n has a nonempty intersection if (and only if) the union of every $n + 1$ or fewer members of the family is starshaped*. The two theorems of this section extend Breen's result in two different directions.

In our first theorem, starshapedness of the union is weakened to n -connectedness, and open sets are considered as well as closed sets. Like the above Klee-type results, this extension of Helly's theorem is equivalent to Brouwer's fixed point theorem.

Theorem 4. *Let a finite family of convex sets in \mathbb{R}^n be given so that all the sets are closed or all are open, and for each $k \leq n + 1$ the union of every k members of the family is $(k - 2)$ -connected. Then all the members of the family have a point in common.*

Proof. It follows from Theorem 3, case (1), that the intersection of every $n + 1$ members of the family is nonempty, whence the conclusion by Helly's theorem. \square

In our second theorem, convexity of the sets is weakened to starshapedness. The proof combines Theorem 3 with a Krasnosel'skiĭ-type theorem of Kołodziejczyk [13] stating that *if every $n + 1$ members of a finite family of closed sets in \mathbb{R}^n have a starshaped union, then the union of all the members of the family is starshaped*. Theorem 5 extends another result of Breen [3], which is the special case $n = 2$.

Theorem 5. *If every $n + 1$ or fewer members of a finite family of closed sets in \mathbb{R}^n have a starshaped union, then all the members of the family have a point in common.*

Proof. By the above-mentioned theorem of Kołodziejczyk, every union of members of the family is starshaped, so the conclusion follows from Theorem 3, case (1). \square

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