HOLOMORPHIC HELICES IN A COMPLEX SPACE FORM

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Abstract. In a complex space form $M$ we shall investigate a smooth curve $\gamma$ which is generated by a holomorphic Killing vector field $X$ on $M$.

Introduction

In Riemannian Geometry it is interesting to investigate “simple curves” in a certain sense. From this point of view some geometers studied Submanifold Theory (for example, see [1], [2]). In this paper, we consider “simple curves” in a complex space form.

Let $M$ be an $n$-dimensional Kähler manifold with complex structure $J$ and Riemannian metric $\langle \cdot, \cdot \rangle$. For a helix $\gamma$ on $M$ of order $d$ ($\leq 2n$) with the associated Frenet frame $\{V_1, \ldots, V_d\}$, we define its complex torsions by $\tau_{ij}(t) = \langle V_i(t), J V_j(t) \rangle$ for $1 \leq i < j \leq d$. In the study of helices in a Kähler manifold their complex torsions play an important role. We shall call $\gamma$ a holomorphic helix if all the complex torsions are constant. Ohnita and the first-named author proved in [3] that a smooth curve $\gamma$ on a complex space form is a holomorphic helix if and only if it is generated by a holomorphic Killing vector field $X$. This is the complex version of the well-known fact that a smooth curve on a real space form is a helix if and only if it is generated by a Killing vector field. Study of holomorphic helices is one of the most interesting objects in differential geometry in a complex space form.

The main purpose of this paper is to study the moduli of holomorphic helices of order 3 in complex space forms. A helix of order 1 is nothing but a geodesic, and a helix of order 2 is called a circle. They are necessarily holomorphic helices. But in the class of helices of order $d$ ($\geq 3$) we can find many helices which are not holomorphic helices.

We show that the moduli of all holomorphic helices of order 3 on an $n$-dimensional complex space form is parametrized by three real numbers or two real numbers according as $n \geq 3$ or $n = 2$ (see Theorem 5). Moreover, we investigate the moduli of all holomorphic helices in a 2-dimensional complex space form (see Theorems 4, 5).

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1197
1. Complex torsions of holomorphic helices

We shall start by recalling the definition of helices. A smooth curve $\gamma = \gamma(t)$ parametrized by its arc length $t$ is called a helix of proper order $d$ if there exist an orthonormal frame $\{V_1 = \dot{\gamma}, \ldots, V_d\}$ along $\gamma$ and positive constants $k_1, \ldots, k_{d-1}$ which satisfy the system of ordinary differential equations
\[
\nabla_t V_j(t) = -k_{j-1}V_{j-1}(t) + k_j V_{j+1}(t), \quad j = 1, \ldots, d,
\]
where $V_0 = V_{d+1} = 0$ and $\nabla_t$ denotes the covariant differentiation along $\gamma$. The constants $k_j$ ($1 \leq j \leq d-1$) and the orthonormal frame $\{V_1, \ldots, V_d\}$ are called the curvatures and the Frenet frame of $\gamma$, respectively. A curve is called a helix of order $d$ if it is a helix of proper order $r$ ($\leq d$). For a helix of order $d$ which is of proper order $r$ ($\leq d$), we use the convention in (1.1) that $k_j = 0$ ($r \leq j \leq d-1$) and $V_j = 0$ ($r + 1 \leq j \leq d$). Note that every helix is a real-analytic curve on a Kähler manifold $M$.

As a matter of course complex torsions of helices satisfy $|\tau_{ij}(t)| \leq 1$ by their definitions. We first show that complex torsions and curvatures of holomorphic helices have relations. Differentiating all the complex torsions, we find by use of the equation (1.1) that
\[
\frac{d}{dt} \tau_{ij}(t) = -k_{i-1} \tau_{i-1,j}(t) + k_i \tau_{i+1,j}(t) - k_{j-1} \tau_{i,j-1}(t) + k_j \tau_{i,j+1}(t),
\]
where $\tau_{kl} = 0$ when $k = l$ or $k = 0$ or $l$ is greater than the proper order. We hence get the following.

Proposition 1. The complex torsions of a holomorphic helix of odd proper order $d$ on a Kähler manifold satisfy the following relations:
\[
\tau_{i,i+2k} = 0 \text{ for } i = 1, 2, \ldots, d - 2k, \text{ where } k = 1, 2, \ldots, (d-1)/2,
\]
\[
k_1 \tau_{2d} = k_{d-1} \tau_{1,d-1},
\]
\[
k_1 \tau_{2j} + k_j \tau_{1,j+1} = k_{j-1} \tau_{1,j-1} \text{ for } j = 3, 5, \ldots, d - 2,
\]
\[
k_{i-1} \tau_{i-1,d} + k_{d-1} \tau_{i,d-1} = k_i \tau_{i+1,d} \text{ for } i = 3, 5, \ldots, d - 2,
\]
\[
k_{i-1} \tau_{i-1,j} + k_{j-1} \tau_{i,j-1} = k_i \tau_{i+1,j} + k_j \tau_{i,j+1} \text{ for } i = 2, 3, \ldots, d - 3, \quad j = i + 2, i + 4, \ldots, d - 1.
\]

Proposition 2. The complex torsions of a holomorphic helix of even proper order $d$ on a Kähler manifold satisfy the following relations:
\[
\tau_{i,i+2k} = 0 \text{ for } i = 1, 2, \ldots, d - 2k, \text{ where } k = 1, 2, \ldots, (d-2)/2,
\]
\[
k_1 \tau_{2d} = k_{d-1} \tau_{1,d-1},
\]
\[
k_1 \tau_{2j} + k_j \tau_{1,j+1} = k_{j-1} \tau_{1,j-1} \text{ for } j = 3, 5, \ldots, d - 1,
\]
\[
k_{i-1} \tau_{i-1,d} + k_{d-1} \tau_{i,d-1} = k_i \tau_{i+1,d} \text{ for } i = 2, 4, \ldots, d - 2,
\]
\[
k_{i-1} \tau_{i-1,j} + k_{j-1} \tau_{i,j-1} = k_i \tau_{i+1,j} + k_j \tau_{i,j+1} \text{ for } i = 2, 3, \ldots, d - 3, \quad j = i + 2, i + 4, \ldots, d - 1.
\]

Conversely, if the Frenet frame of a helix $\gamma$ in a Kähler manifold satisfies the above relations at a point, then all $n$th derivatives of its complex torsions vanish at this point. Since $\gamma$ is real analytic, we find that it is a holomorphic helix. We therefore have
Proposition 3. For orthonormal vectors $v_1, \ldots, v_d$ at a point $p$ of a Kähler manifold $M$, we set $\tau_{ij} = \langle v_i, Jv_j \rangle$ ($1 \leq i < j \leq d$). If positive constants $k_1, \ldots, k_{d-1}$ and the vectors $v_1, \ldots, v_d$ satisfy the relations in Proposition 1 or 2, there exists a unique holomorphic helix with curvatures $k_1, \ldots, k_{d-1}$ satisfying that the initial value of its Frenet frame is $(v_1, \ldots, v_d)$.

The following is easily verified.

Proposition 4. The complex torsions $\tau_{ij}$ of a holomorphic helix of proper order $d$ on a Kähler manifold $M$ satisfy $\sum_{j=1}^{l-1} \tau_{ji}^2 + \sum_{j=l+1}^{d} \tau_{ji}^2 \leq 1$ for every $i$.

Proof. Since $\tau_{i,i+2k} = 0$, we find that $\{V_{2l-1}, JV_{2l-1}|l = 1, 2, \ldots\}$ and $\{V_{2l}, JV_{2l}|l = 1, 2, \ldots\}$ form orthonormal frames. When $i$ is even, the left-hand-side of the inequality is the norm of the projection of $V_i(0)$ onto the complex linear subspace spanned by $\{V_{2l-1}(0), JV_{2l-1}(0)|l = 1, 2, \ldots\}$. We hence get the inequality. For odd $i$ we have a similar argument.

Here we treat holomorphic helices of order 3. We need to choose orthonormal vectors $v_1, v_2, v_3 \in T_p M$ which satisfy

$$k_1 \langle v_2, Jv_3 \rangle = k_2 \langle v_1, Jv_2 \rangle, \langle v_1, Jv_3 \rangle = 0.$$

Identifying $T_p M$ with $\mathbb{C}^3$, we set $v_1, v_2$ and $v_3$ as

$$v_1 = (1, 0, \ldots, 0),$$

$$v_2 = (-i\tau, \sqrt{1 - \tau^2}, 0, \ldots, 0),$$

$$v_3 = (0, -i\rho/\sqrt{1 - \tau^2}, \sqrt{1 - \tau^2 - \rho^2}/\sqrt{1 - \tau^2}, 0, \ldots, 0)$$

for positive constants $\tau$ and $\rho$ with $\tau^2 + \rho^2 \leq 1$. Then they are orthonormal and satisfy $\langle v_1, Jv_2 \rangle = \tau, \langle v_2, Jv_3 \rangle = \rho, \langle v_1, Jv_3 \rangle = 0$. We therefore have

Theorem 1. Let $M$ be a Kähler manifold of dimension greater than 2. Then the following hold:

1. Every holomorphic helix of order 3 satisfies

$$k_1 \tau_{23} = k_2 \tau_{12}, \tau_{13} = 0, |\tau_{12}| \leq k_1/\sqrt{k_1^2 + k_2^2}.$$

2. Conversely, if nonnegative constants $k_1, k_2$ and a constant $\tau$ satisfy $|\tau| \leq k_1/\sqrt{k_1^2 + k_2^2}$, then there exists a holomorphic helix of order 3 on $M$ with the first curvature $k_1$ and the second curvature $k_2$, and with the first complex torsion $\tau_{12} = \tau$.

3. If $|\tau| > k_1/\sqrt{k_1^2 + k_2^2}$, we have no such a holomorphic helix of order 3 on $M$.

Theorem 2. Let $M$ be a 2-dimensional Kähler manifold. Then the following hold:

1. The complex torsions of each holomorphic helix of proper order 3 in $M$ are

$$\tau_{12} = k_1/\sqrt{k_1^2 + k_2^2}, \tau_{13} = 0, \tau_{23} = k_2/\sqrt{k_1^2 + k_2^2}$$

or

$$\tau_{12} = -k_1/\sqrt{k_1^2 + k_2^2}, \tau_{13} = 0, \tau_{23} = -k_2/\sqrt{k_1^2 + k_2^2},$$

where its curvatures are $k_1$ and $k_2$.

2. Conversely for given positive constants $k_1$ and $k_2$, there exists a holomorphic helix of proper order 3 with curvatures $k_1$ and $k_2$, and with complex torsions defined by (1.2) or (1.3).
Such a description as above for holomorphic helices of order 4 is much more complicated. We restrict ourselves here to holomorphic helices on a 2-dimensional Kähler manifold $M$. For given constants $\tau$ and $\rho$ with $\tau^2 + \rho^2 = 1$, we choose vectors

$$v_1 = (1, 0), \quad v_2 = (-i\tau, \rho), \quad v_3 = (0, -i), \quad v_4 = \mp(i\rho, \tau)$$

in $T_p M \simeq C^2$. Then they are orthonormal and satisfy

$$\langle v_1, Jv_2 \rangle = \tau, \quad \langle v_2, Jv_3 \rangle = \rho, \quad \langle v_1, Jv_4 \rangle = \pm \rho \quad \text{and} \quad \langle v_2, Jv_4 \rangle = \pm \tau.$$ 

On the other hand, Proposition 2 shows that a helix is a holomorphic helix if and only if

$$\tau_{13}(0) = \tau_{24}(0) = 0, \quad k_1\tau_{23}(0) + k_3\tau_{14} = k_2\tau_{12}(0),$$

$$k_1\tau_{14}(0) + k_3\tau_{25}(0) = k_2\tau_{34}(0).$$

We therefore have

**Theorem 3.** Let $M$ be a 2-dimensional Kähler manifold. Then the following hold:

1. The complex torsions of each holomorphic helix of proper order 4 with curvatures $k_1, k_2$ and $k_3$ on $M$ satisfy one of the following:

   (1.4) \hspace{1cm} \tau_{12} = \tau_{34} = \tau, \quad \tau_{23} = \tau_{14} = k_2\tau/(k_1 + k_3), \quad \tau_{13} = \tau_{24} = 0,$

   where $\tau = \pm(k_1 + k_3)/\sqrt{k_2^2 + (k_1 + k_3)^2}$,

   (1.5) \hspace{1cm} \tau_{12} = -\tau_{34} = \tau, \quad \tau_{23} = -\tau_{14} = k_2\tau/(k_1 - k_3), \quad \tau_{13} = \tau_{24} = 0,$

   when $k_1 \neq k_3$, where $\tau = \pm(k_1 - k_3)/\sqrt{k_2^2 + (k_1 - k_3)^2}$, or

   (1.5′) \hspace{1cm} \tau_{12} = \tau_{34} = \tau_{13} = \tau_{24} = 0, \quad \tau_{23} = -\tau_{14} = \pm 1,$

   when $k_1 = k_3$.

2. Conversely, for given positive constants $k_1, k_2$ and $k_3$, there exist holomorphic helices of proper order 4 in $M$ with curvatures $k_1, k_2$ and $k_3$, and with complex torsions defined by (1.4), (1.5) or (1.4), (1.5′).

2. Moduli of holomorphic helices on a complex space form

Let $M_n(c)$ be an $n$-dimensional complete simply connected complex space form of constant holomorphic sectional curvature $c$. It is well-known that an arbitrary complex space form is locally complex analytically isometric to a complex projective space, a complex hyperbolic space or a complex Euclidean space according as the holomorphic sectional curvature $c$ is positive, negative or zero. We have the following holomorphic congruence theorem for helices in $M_n(c)$.

**Proposition 5** ([3]). Let $\gamma$ and $\sigma$ be two helices of orders $p$ and $q$ in a complex space form $M_n(c)$, respectively. Let $\{k_1, \ldots, k_{p-1}\}$ (resp. $\{\lambda_1, \ldots, \lambda_{q-1}\}$) be the curvatures of $\gamma$ (resp. $\sigma$), and let $\tau_{ij}^\gamma(t)$ (resp. $\tau_{ij}^\sigma(t)$) be the complex torsions of $\gamma$ (resp. $\sigma$). Then there exists a holomorphic isometry $\varphi$ of $M_n(c)$ satisfying $\gamma = \varphi \circ \sigma$ if and only if $p = q$, $k_i = \lambda_i$ $\left(1 \leq i \leq p-1\right)$ and $\tau_{ij}^\gamma(0) = \tau_{ij}^\sigma(0)$ $\left(1 \leq i < j \leq p\right)$.

In this section we denote by $HH^d(M_n(c))$ the set of the equivalence classes of all holomorphic helices of order $d$ ($\leq 2n$) in $M_n(c)$ with respect to holomorphic isometries of $M_n(c)$. By Proposition 5 the set $HH^d(M_n(c))$ is naturally regarded as a subset of $[0, \infty)^{d-1} \times [-1, 1]^{d(d-1)/2} \subset R^{(d+2)(d-1)/2}$. Needless to say, every
holomorphic helix which lies on a totally real totally geodesic submanifold $M^n(c/4)$ of $M_n(c)$ is a holomorphic helix (whose complex torsions are zero in $M_n(c)$), so that the set of the equivalence classes of all holomorphic helices of proper order $d$ ($\leq n$) with respect to holomorphic isometries of $M_n(c)$, which is a subset of $Hh^d(M_n(c))$, is not empty.

**Theorem 4.** For given positive constants $k_1, k_2$ and $k_3$, there exist four equivalence classes of holomorphic helices of proper order 4 with curvatures $k_1, k_2$ and $k_3$ with respect to holomorphic isometries of $M_2(c)$. In addition, these four equivalence classes are given by (1.4), (1.5) or (1.4), (1.5').

We give some examples here of holomorphic helices of proper order 4 in a 2-dimensional complex projective space $CP_2(c)$. Let $\pi : S^{2n+1}(1)(\subset C^{n+1}) \to CP^n(4)$ denote the Hopf fibration.

**Example 1.** For any $k$ satisfying $0 < k < \sqrt{2}$, we put

$$A = \sqrt{(4 - k^2 - \sqrt{(8 - k^2)(8 - k^2)})/(2(8 - k^2))},$$

$$B = 2/\sqrt{8 - k^2},$$

$$C = \sqrt{(4 - k^2 + \sqrt{(8 - k^2)(8 - k^2)})/(2(8 - k^2))},$$

$$\alpha = (\sqrt{2 - k^2 + \sqrt{8 - k^2}})/\sqrt{2},$$

$$\beta = (\sqrt{2 - k^2}/\sqrt{2}),$$

$$\delta = \left(\sqrt{2 - k^2 - \sqrt{8 - k^2}}\right)/\sqrt{2}.$$  

Let $\tilde{\gamma}$ be a curve in $C^3$ defined by $\tilde{\gamma}(t) = (Ae^{\alpha t}, Be^{\beta t}, Ce^{\gamma t})$. Then $\tilde{\gamma}$ is a horizontal curve on $S^3(1)$ parametrized by arc length $t$. Moreover $\pi(\tilde{\gamma})$ is a holomorphic helix of order 4 in $CP_2(4)$ with curvatures $k_1 = k$, $k_2 = \sqrt{(18 - 9k^2)/2}$ and $k_3 = k$, and with complex torsions $\tau_{12} = \tau_{13} = \tau_{24} = \tau_{34} = 0$, $\tau_{14} = 1$ and $\tau_{23} = -1$. The curve $\pi(\tilde{\gamma})$ satisfies (1.5').

**Example 2.** Let $\tilde{\gamma}$ be a curve in $C^3$ defined by

$$\tilde{\gamma}(t) = \left((1/\sqrt{3}) e^{3t}, (1/\sqrt{14}) e^{2t}, (5/\sqrt{42}) e^{-4t/5}\right).$$

Then $\pi(\tilde{\gamma})$ is a holomorphic helix of order 4 in $CP_2(4)$ with curvatures $k_1 = 3\sqrt{2}/5$, $k_2 = 11\sqrt{2}/10$ and $k_3 = 1/\sqrt{2}$, and with complex torsions $\tau_{12} = \tau_{14} = \tau_{23} = \tau_{34} = -1/\sqrt{2}$ and $\tau_{13} = \tau_{24} = 0$. The curve $\pi(\tilde{\gamma})$ satisfies (1.4).

Finally we shall investigate the moduli spaces $Hh^d(M_n(c))$ ($d = 1, 2, 3$). The moduli space $Hh^1(M_n(c))$ clearly consists of one point. As an immediate consequence of Theorem 1, Theorem 2 and Proposition 5 we can establish the following.

**Theorem 5.** (1) The moduli space $Hh^2(M_n(c))$ is homeomorphic to a cone in $R^2$ or a half line according as $n \geq 2$ or $n = 1$. More precisely, $Hh^2(M_n(c))$ is $[0, \infty) \times [-1, 1]/\sim$ or $[0, \infty)$ according as $n \geq 2$ or $n = 1$, where the equivalence relation $\sim$ means that $(0, \tau) \sim (0, \rho)$ if $\tau, \rho \in [-1, 1]$. 

(2) The moduli space $Hh^3(M_n(c))$ is connected and

$$Hh^3(M_n(c)) = \begin{cases} (k_1, k_2, \tau) \in [0, \infty) \times [0, \infty) \times [-1, 1] | \tau^2 \leq k_1^2/(k_1^2 + k_2^2) \} / \sim, & n \geq 3, \\
(0, \infty) \times \{0\} \times [-1, 1] \cup \{ (k_1, k_2, \pm k_1/\sqrt{k_1^2 + k_2^2} | k_1 > 0, k_2 > 0 \} / \sim, & n = 2,
\end{cases}$$

where the equivalence relation $\sim$ means that $(0, k, \tau) \sim (0, l, \rho)$ if $k, l \in [0, \infty)$ and $\tau, \rho \in [-1, 1]$.

Remark. Let $\gamma$ be a holomorphic helix of proper order 3 with curvatures $k_1$ and $k_2$, and with the first complex torsion $\tau_1 = \tau$ in a complex space form $M_n(c)$. Then $\gamma$ lies on a totally real totally geodesic submanifold $M^3(c/4)$ of $M_n(c)$ if and only if $\tau = 0$, and $\gamma$ lies on a holomorphic totally geodesic submanifold $M_2(c)$ of $M_n(c)$ if and only if $\tau = \pm k_1/\sqrt{k_1^2 + k_2^2}$.

References


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