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ON AN OPTIMALITY PROPERTY OF RAMANUJAN SUMS

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Abstract. We evaluate $\inf_{b_n} \sum_{a=1}^{q} |\sum_{\substack{n=1\\(n,q)=1}}^{q} b_n e^{2\pi i a n/q}|$, where the inf is taken over sequences b_n satisfying $b_n \geq 1$. In particular we show that it is attained by taking $b_n = 1$ for all n, which reduces the summation over n to a Ramanujan sum $c_q(a) = \sum_{\substack{n=1\\(n,q)=1}}^{q} e^{2\pi i a n/q}$.

Let a positive integer q be fixed. In this note we consider the problem of determining

$$\inf_{b_n} \sum_{a=1}^{q} \left| {\sum_n}' b_n e\left(\frac{an}{q}\right) \right|, \tag{1}$$

where $e(\alpha)$ stands for $e^{2\pi i \alpha}$, $\sum_n'$ denotes the summation over n in the range $1 \leq n \leq q$, $(n,q) = 1$, and the infimum is taken over all sequences b_n satisfying

$$b_n \geq 1. \tag{2}$$

We note that if $b_n = 1$ for all n, the innermost sum in (1) is the Ramanujan sum

$$c_q(a) = {\sum_n}' e\left(\frac{an}{q}\right).$$

Using the well-known identity

$$c_q(a) = \frac{\varphi(q)\,\mu\left(q/(a,q)\right)}{\varphi\left(q/(a,q)\right)}, \tag{3}$$

where φ and μ are Euler's and Möbius functions respectively (see, for example, [HW, Theorem 272]), and letting $q_0 = \prod_{p|q} p$ be the square-free kernel of q, we obtain

$$\begin{aligned} \sum_{a=1}^{q} |c_q(a)| &= \sum_{a=1}^{q} \frac{\varphi(q)\mu^2\left(q/(a,q)\right)}{\varphi\left(q/(a,q)\right)} \\ &= \varphi(q) \sum_{d|q_0} \frac{1}{\varphi(d)} \sum_{\substack{1\leq a\leq q\\(a,q)=q/d}} 1 = \varphi(q) \sum_{d|q_0} 1 \\ &= \varphi(q) 2^{\omega(q)}, \end{aligned} \tag{4}$$

where $\omega(q)$ is the number of distinct prime divisors of q. Thus (4) gives a "trivial" upper bound for (1). In fact this author was led to consider this problem while

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working on estimates for more general exponential sums in an attempt to beat this estimate by introducing a "smoothing factor" b_n which had to satisfy (2). This might seem plausible at first since if $q/(a,q) > 3$, one can easily find b_n satisfying (2) for which

$$\sum\nolimits_n' b_n e\left(\frac{an}{q}\right) = 0.$$

We will show however that (1) is attained by taking $b_n = 1$ for all n. In fact, the following more general result is no more difficult.

Theorem. *Let r be a real number satisfying $r \geq 1$. Then for any sequence of complex numbers b_n we have*

$$\sum_{a=1}^{q} \left|\sum\nolimits_n' b_n e\left(\frac{an}{q}\right)\right|^r \geq \left(\frac{\left|\sum_n' b_n\right|}{\varphi(q)}\right)^r \sum_{a=1}^{q} |c_q(a)|^r . \tag{5}$$

Proof. We may assume that b_n is defined for all integers n and is periodic with period q. We set

$$B = \sum\nolimits_n' b_n$$

and

$$S = \sum_{a=1}^{q} \left|\sum\nolimits_n' b_n e\left(\frac{an}{q}\right)\right|^r .$$

Using n^* to denote the multiplicative inverse of n modulo q for $(n,q) = 1$, we write

$$\begin{aligned} |B|^r \sum_{a=1}^{q} |c_q(a)|^r &= \sum_{a=1}^{q} |Bc_q(a)|^r \\ &= \sum_{a=1}^{q} \left|\sum\nolimits_m' \left(\sum\nolimits_n' b_{mn}\right) e\left(\frac{am}{q}\right)\right|^r \\ &= \sum_{a=1}^{q} \left|\sum\nolimits_n' \sum\nolimits_m' b_{mn} e\left(\frac{an^*mn}{q}\right)\right|^r . \end{aligned}$$

By the Hölder inequality, or trivially in the case $r = 1$, the last summation over n is bounded by

$$\leq \varphi(q)^{1/r'} \left(\sum\nolimits_n' \left|\sum\nolimits_m' b_{mn} e\left(\frac{an^*mn}{q}\right)\right|^r\right)^{1/r},$$

where r' satisfies $1/r + 1/r' = 1$. Therefore

$$\begin{aligned} |B|^r \sum_{a=1}^{q} |c_q(a)|^r &\leq \varphi(q)^{r/r'} \sum\nolimits_n' \sum_{a=1}^{q} \left|\sum\nolimits_m' b_{mn} e\left(\frac{an^*mn}{q}\right)\right|^r \\ &= \varphi(q)^{r/r'+1} S \\ &= \varphi(q)^r S, \end{aligned}$$

and the theorem follows. □

We observe that (5) may fail for $r < 1$. For example, taking $q = 5$, $b_1 = b_4 = 1 + 1/(2\cos(2\pi/5))$ and $b_2 = b_3 = 1$, we obtain, by (3),

$$\begin{aligned}\sum_{a=1}^{5}\left|\sum_{n=1}^{4} b_n e\left(\frac{an}{5}\right)\right|^r &= \sum_{a=1}^{4}\left|\frac{1}{2\cos(2\pi/5)}\left(e\left(\frac{a}{5}\right)+e\left(\frac{4a}{5}\right)\right)-1\right|^r + \left|4+\frac{1}{\cos(2\pi/5)}\right|^r \\ &= 2\left|\frac{\cos(4\pi/5)}{\cos(2\pi/5)}-1\right|^r + \left|4+\frac{1}{\cos(2\pi/5)}\right|^r,\end{aligned} \tag{6}$$

and

$$\sum_{a=1}^{5}|c_q(a)|^r = 4 + 4^r. \tag{7}$$

Thus for $r > 0$ sufficiently small (6) will be smaller than (7).

References

[HW] G.H. Hardy & E.M. Wright, *An introduction to the theory of numbers*, fifth edition, Oxford University Press, 1979. MR **81i:**10002

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