HYPERSURFACES IN A SPHERE
WITH CONSTANT MEAN CURVATURE

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Abstract. Let $M^n$ be a closed hypersurface of constant mean curvature immersed in the unit sphere $S^{n+1}$. Denote by $S$ the square of the length of its second fundamental form. If $S < 2\sqrt{n-1}$, $M$ is a small hypersphere in $S^{n+1}$. We also characterize all $M^n$ with $S = 2\sqrt{n-1}$.

1. Introduction

Let $M^n$ be a closed submanifold with parallel mean curvature vector field immersed in the unit sphere $S^{n+p}$. Denote by $H$ the length of the mean curvature vector field and by $S$ the square of the length of the second fundamental form of $M^n$. It is important to characterize those $M^n$ immersed as $n$-spheres in $S^{n+p}$ by $H$ and $S$.

When $M$ is minimal, J. Simons [9] obtained a pinching constant $n/(2 - 1/p)$ of $S$ and Chern-do Carmo-Kobayashi [3] showed that it is sharp and characterized all $M$ with $S = n/(2 - 1/p)$. M. Okumura [6, 7] first discussed the general case and gave a pinching constant of $S$, but it is not sharp. Recently the sharp ones were obtained by H. Alencar-M. do Carmo [1] for $p = 1$, W. Santos [8] for $p > 1$ and H. W. Xu [11] for $p \geq 1$ respectively. But all of them were expressed by the mean curvature $H$. S. T. Yau [12] obtained a pinching constant for $p > 1$ which depended only on $n$ and $p$. H. W. Xu [10] improved Yau’s result, but far from sharpness.

In the present paper, we shall give a pinching constant for $p = 1$ which depends only on $n$ and show the sharpness of it. More precisely, we want to prove the following theorems:

Theorem A. Let $M^n$ be a hypersurface of constant mean curvature immersed in $S^{n+1}$ with constant length of the second fundamental form. Then:

1. If $S < 2\sqrt{n-1}$, $M^n$ is locally a piece of small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n + S)}$.
2. If $S = 2\sqrt{n-1}$, $M^n$ is locally a piece of either $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$ where $r_0^2 = n/(n + 2\sqrt{n-1})$, $r^2 = 1/(\sqrt{n-1} + 1)$ and $s^2 = \sqrt{n-1}/(\sqrt{n-1} + 1)$.

Theorem A’. Let $M^n$ be a closed hypersurface of constant mean curvature immersed in $S^{n+1}$. Then:

1. If $S < 2\sqrt{n-1}$, $M^n$ is a small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n + S)}$. 
Substituting this into (1), we have
\[ \sum_{(i,j)} h_{ij} \Delta h_{ij} = nS + nHf - n^2H^2 - S^2, \]
where \( f = \text{Tr} L^3 \) (cf. e.g. [2] or [7]).

M. Okumura [7] established the following lemma (see also [1] or [11]).

**Lemma.** Let \( \{a_i\}_{i=1}^n \) be a set of real numbers satisfying \( \sum_{(i)} a_i = 0, \sum_{(i)} a_i^2 = t^2 \), where \( t \geq 0 \). Then we have

\[ -\frac{n-2}{\sqrt{n(n-1)}} t^3 \leq \sum_{(i)} a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} t^3, \]

and equalities hold if and only if at least \((n-1)\) of the \(a_i\)'s are equal to one another.

Suppose that \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the principal curvatures of \( M \). Then we have

\[ nH = \sum_{(i)} \lambda_i, \quad S = \sum_{(i)} \lambda_i^2, \quad f = \sum_{(i)} \lambda_i^3, \]

Set \( \tilde{S} = S - nH^2 \), \( \tilde{f} = f - 3HS + 2nH^3 \) and \( \tilde{\lambda}_i = \lambda_i - H \) \((1 \leq i \leq n)\). Then (3) changes into

\[ 0 = \sum_{(i)} \tilde{\lambda}_i, \quad \tilde{S} = \sum_{(i)} \tilde{\lambda}_i^2, \quad \tilde{f} = \sum_{(i)} \tilde{\lambda}_i^3. \]

By applying Okumura's Lemma to \( \tilde{f} \) in (4), we have

\[ \tilde{f} \geq -\frac{n-2}{\sqrt{n(n-1)}} \tilde{S} \sqrt{\tilde{S}} \iff f \geq 3HS - 2nH^3 - \frac{n-2}{\sqrt{n(n-1)}} S \sqrt{S}. \]

Substituting this into (1), we have

\[ \sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \tilde{S} \left\{ n - (\tilde{S} - nH^2) - (n-2)H \sqrt{\frac{n}{n-1} \tilde{S}} \right\}. \]

Consider the quadratic form \( Q(u, t) = u^2 - \frac{n-2}{\sqrt{n-1}} ut - t^2 \). By the orthogonal transformation

\[
\begin{align*}
\tilde{u} &= \frac{1}{\sqrt{2n}} \left\{ (1 + \sqrt{n-1})u + (1 - \sqrt{n-1})t \right\}, \\
\tilde{t} &= \frac{1}{\sqrt{2n}} \left\{ (\sqrt{n-1})u + (\sqrt{n-1} + 1)t \right\},
\end{align*}
\]

2. **Proof of the Theorems**

Let \( M \) be a closed hypersurface immersed in the unit sphere \( S^{n+1} \). Take a local orthonormal coframe field \( \{\omega_i\}_{i=1}^n \) on \( M \). Then the second fundamental form can be expressed as \( L = (h_{ij})_{n \times n} \). The mean curvature \( H \) and the square of the length of the second fundamental form \( S \) are defined by \( H = \frac{1}{n} \sum_{(i)} h_{ii}, S = \sum_{(i,j)} (h_{ij})^2 \).

From now on, we shall always use \( i, j, k, \ldots \) for indices running from 1 to \( n \).

Denote the covariant differentials of \( \{h_{ij}\} \) by \( \{h_{ijk}\} \) and \( \{h_{ijkl}\} \). Then the Laplacian of \( h_{ij} \) is defined by \( \Delta h_{ij} = \sum_{(k)} h_{ijkk} \). It follows that

\[ \sum_{(i,j)} h_{ij} \Delta h_{ij} = nS + nHf - n^2H^2 - S^2, \]
\[\text{where } f = \text{Tr} L^3 \text{ (cf. e.g. [2] or [7]).} \]

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Q(u, t) turns into \(Q(u, t) = \frac{n}{2\sqrt{n-1}}(\tilde{u}^2 - \tilde{t}^2)\), where \(\tilde{u}^2 + \tilde{t}^2 = u^2 + t^2 = S\).

Take \(t = \sqrt{S}\) and \(u = \sqrt{nH}\) in \(Q(u, t)\), and substitute it into (5). We can see

\[
\sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \bar{S} \left(n - \frac{n}{2\sqrt{n-1}} S + \frac{n}{\sqrt{n-1}} \tilde{u}^2\right) \geq \bar{S} \left(n - \frac{n}{2\sqrt{n-1}} S\right).
\]

Therefore we have

\[
\frac{1}{2} \Delta S = \sum_{(i,j,k)} h_{ijk}^2 + \sum_{(i,j)} h_{ij} \Delta h_{ij} \geq \bar{S} \left(n - \frac{n}{2\sqrt{n-1}} S\right).
\]

**Theorem A.** Let \(M^n\) be a hypersurface of constant mean curvature immersed in \(S^{n+1}\) with constant length of the second fundamental form. Then:

1. If \(S < 2\sqrt{n-1}\), \(M\) is locally a piece of a small hypersphere \(S^n(r)\) in \(S^{n+1}\), where \(r = \sqrt{n/(n+S)}\).
2. If \(S = 2\sqrt{n-1}\), \(M\) is locally a piece of either \(S^n(r_0)\) or \(S^1(r) \times S^{n-1}(s)\), where \(r_0^2 = n/(n+2\sqrt{n-1})\), \(r^2 = 1/((\sqrt{n-1}) + 1)\) and \(s^2 = \sqrt{n-1}/((\sqrt{n-1}) + 1)\).

**Proof.** Since \(S\) is constant, the left-hand side of (7) is zero. When \(S \leq 2\sqrt{n-1}\), we have

\[
\bar{S} \left(n - \frac{n}{2\sqrt{n-1}} S\right) = 0, \quad h_{ijk} = 0, \quad 1 \leq i, j, k \leq n.
\]

If \(S < 2\sqrt{n-1}\), we have \(\bar{S} = 0\), which means that \(M\) is totally umbilical and hence is locally a piece of hypersphere \(S^n(r)\) where \(r = \sqrt{n/(n+S)}\).

Suppose \(S = 2\sqrt{n-1}\). Then all of the inequalities in (5)–(7) become equal ones. Okumura’s Lemma implies that at least \((n-1)\) of \(\lambda_i\)’s are equal to one another. When \(\lambda_1 = \lambda_2 = \cdots = \lambda_n\), \(M\) is totally umbilical and hence is locally a piece of hypersphere \(S^n(r)\) where \(r^2 = n/(n+2\sqrt{n-1})\). When \(M\) is not totally umbilical, there are exactly \((n-1)\) of \(\lambda_i\)’s that are equal to one another. The same arguments as those developed by Chern-do Carmo-Kobayashi (see [3], p. 68) show that \(M\) is locally a piece of \(S^1(r) \times S^{n-1}(s)\) in \(S^{n+1}\). To determine the radii \(r\) and \(s\), we refer to the examples of K. Nomizu and B. Smyth [5], from which we have

\[
H = -\frac{1}{n} \left(\frac{s}{r}\right) + \frac{n-1}{n} \left(\frac{r}{s}\right), \quad S = \left(\frac{s}{r}\right)^2 + (n-1) \left(\frac{r}{s}\right)^2.
\]

It is easy to see that

\[
\left(\frac{s}{r}\right)^2 + (n-1) \left(\frac{r}{s}\right)^2 \geq 2\sqrt{n-1}
\]

and equality holds if and only if \(\left(\frac{s}{r}\right)^2 = \sqrt{n-1}\). Therefore we have \(r^2 = \frac{1}{\sqrt{n-1}+1}\) and \(s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}\).

When \(M\) is closed, the integral of the left-hand side of (7) on \(M\) is equal to zero, and so is that of the right-hand side. After the same deduction as in the proof of Theorem A, we can obtain the following:

**Theorem A’.** Suppose \(M\) is a closed hypersurface of constant mean curvature immersed in \(S^{n+1}\). Then:

1. If \(S < 2\sqrt{n-1}\), \(M\) is a small hypersphere \(S^n(r)\), where \(r = \sqrt{n/(n+S)}\).
2. If \(S = 2\sqrt{n-1}\), \(M\) is either a small hypersphere \(S^n(r_0)\) or \(S^1(r) \times S^{n-1}(s)\), where \(r_0, r\) and \(s\) are taken as in Theorem A.
We can show an application of Theorem A'. H. W. Xu [10] proved the following:

**Proposition (Xu).** Let $M^n$ be an $n$-dimensional compact submanifold with parallel mean curvature vector field in $S^{n+p}$ and $p > 1$. If

$$S \leq \min \left\{ \frac{2n}{1 + \sqrt{n}}, \frac{n}{2 - (p - 1)^{-1}} \right\},$$

and the Gauss mapping of $M$ is relatively affine, then $M^n$ is a standard hypersphere in a totally geodesic $S^{n+1}$ of $S^{n+p}$.

By Theorem A', we can remove the assumption that the Gauss mapping is relatively affine. Namely we can obtain the following

**Corollary.** Let $M^n$ be an $n$-dimensional compact submanifold with parallel mean curvature vector field in $S^{n+p}$ and $p > 1$. If

$$S \leq \min \left\{ \frac{2n}{1 + \sqrt{n}}, \frac{n}{2 - (p - 1)^{-1}} \right\},$$

then $M^n$ is a standard hypersphere in a totally geodesic $S^{n+1}$ of $S^{n+p}$.

**Proof.** It is easy to check that $(\sqrt{n} + 1)/n > 1/\sqrt{n} - 1$. Therefore we have

$$\frac{n}{\sqrt{n} - 1} > \frac{n}{\sqrt{n} + 1} \iff 2\sqrt{n} - 1 > \frac{2n}{\sqrt{n} + 1} \geq S.$$

\[\square\]

**References**


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