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ON PERFECT SIMPLE-INJECTIVE RINGS

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Dedicated to Professor K. Varadarajan on the occasion of his sixtieth birthday

Abstract. Harada calls a ring R right simple-injective if every R-homomorphism with simple image from a right ideal of R to R is given by left multiplication by an element of R. In this paper we show that every left perfect, left and right simple-injective ring is quasi-Frobenius, extending a well known result of Osofsky on self-injective rings. It is also shown that if R is left perfect and right simple-injective, then R is quasi-Frobenius if and only if the second socle of R is countably generated as a left R-module, extending many recent results on self-injective rings. Examples are given to show that our results are non-trivial extensions of those on self-injective rings.

A ring R is called quasi-Frobenius if R is left (and right) artinian and left (and right) self-injective. A well known result of Osofsky [15] asserts that a left perfect, left and right self-injective ring is quasi-Frobenius. It has been conjectured by Faith [9] that a left (or right) perfect, right self-injective ring is quasi-Frobenius. This conjecture remains open even for semiprimary rings.

Throughout this paper all rings R considered are associative with unity and all modules are unitary R-modules. We write M_R to indicate a right R-module. The socle of a module is denoted by $\mathrm{soc}(M)$. We write $N \subseteq M$ ($N \subseteq^{\mathrm{ess}} M$) to mean that N is a submodule (essential) of M. For any subset X of R, $l(X)$ and $r(X)$ denote, respectively, the left and right annihilators of X in R.

A ring R is called right **Kasch** if every simple right R-module is isomorphic to a minimal right ideal of R. The ring R is called right **pseudo-Frobenius** (a right **PF-ring**) if R_R is an injective cogenerator in mod-R; equivalently if R is semiperfect, right self-injective and has an essential right socle.

A ring R is called right **principally injective** if every R-morphism from a principal right ideal of R into R is given by left multiplication. In [14], a ring R is called a right **generalized pseudo-Frobenius** ring (a right **GPF-ring**) if R is semiperfect, right principally injective and has an essential right socle.

We write $J = J(R)$ for the Jacobson radical of the ring R. Following Fuller [10], if R is semiperfect with a basic set E of primitive idempotents, and if $e, f \in E$, we say that the pair (eR, Rf) is an **i-pair** if $\mathrm{soc}(eR) \cong fR/fJ$ and $\mathrm{soc}(Rf) \cong Re/Je$.

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Let M_R and N_R be R-modules. Following Harada [12], M is said to be a **simple-N-injective module** if, for any submodule $X \subseteq N$ and any R-morphism $\gamma\colon X \to M$ such that $\mathrm{im}(\gamma)$ is simple, there exists an R-morphism $\hat{\gamma}\colon N \to M$ such that $\hat{\gamma}|_X = \gamma$. We call a ring R right **simple-injective** if R_R is simple-R-injective; equivalently, if I is a right ideal of R and $\gamma\colon I \to R$ is an R-morphism with simple image, then $\gamma = c\cdot$ is left multiplication by an element $c \in R$.

If M is a right R-module, $\mathrm{soc}_\alpha(M)$ is defined for each ordinal α as follows: (1) $\mathrm{soc}_1(M) = \mathrm{soc}(M)$; (2) If $\mathrm{soc}_\alpha(M)$ has been defined, then $\mathrm{soc}_{\alpha+1}(M)$ is given by $\mathrm{soc}[M/\mathrm{soc}_\alpha(M)] = \mathrm{soc}_{\alpha+1}(M)/\mathrm{soc}_\alpha(M)$; (3) If α is a limit ordinal, then $\mathrm{soc}_\alpha(M) = \bigcup_{\beta<\alpha} \mathrm{soc}_\beta(M)$. The series $\mathrm{soc}_1(M) \subseteq \mathrm{soc}_2(M) \subseteq \cdots$ is called the **Lowey series** of the module M.

Motivated by the work of Armendariz and Park [2] and that of Ara and Park [1], it was shown by Clark and Huynh [8] that if R is a left and right perfect, right self-injective ring, then R is quasi-Frobenius if and only if $\mathrm{soc}_2(R)$ is finitely generated as a right R-module. The authors did not indicate whether their result remains valid if $\mathrm{soc}_2(R)$ is finitely generated as a left R-module. However it is not difficult to see that if R is left perfect and right self-injective, and if $\mathrm{soc}_2(R)$ is finitely generated as a left R-module, then R is quasi-Frobenius. Indeed, since R is a right PF-ring, $l(J) = r(J) = \mathrm{soc}({}_RR) = \mathrm{soc}(R_R)$, and hence $l(J^2) = r(J^2) = \mathrm{soc}_2({}_RR) = \mathrm{soc}_2(R_R)$. Suppose $\mathrm{soc}_2(R) = \sum_{i=1}^n Ra_i$, $a_i \in R$, and define $\varphi\colon R \to \bigoplus_{i=1}^n a_iR$ by $\varphi(r) = (a_1r, \ldots, a_nr)$. Then $\ker\varphi = r(\mathrm{soc}_2(R)) = rl(J^2) = J^2$. Thus $R/J^2 \hookrightarrow \bigoplus_{i=1}^n a_iR$ and it follows that J/J^2 is right finite dimensional (because R is a right PF-ring). Hence J/J^2 is right finitely generated, and so R is right artinian by a result of Osofsky [15]. This observation implies several results in the literature (see for example [2]) and will be generalized in Theorem 2 of this paper.

The following two lemmas will be needed in our investigation.

Lemma 1 ([14], Theorem 2.3)**.** *Let R be a right GPF-ring. If $\{e_1, \ldots, e_n\}$ is a basic set of primitive idempotents, there exists a (Nakayama) permutation σ of $\{e_1, \ldots, e_n\}$ such that, for each $k = 1, 2, \ldots, n$, $\mathrm{soc}(Re_{\sigma k}) \cong Re_k/Je_k$ is essential in $Re_{\sigma k}$ and $\mathrm{soc}(e_kR)$ is homogeneous with each simple submodule isomorphic to $e_{\sigma k}R/e_{\sigma k}J$. We also have $\mathrm{soc}({}_RR) = \mathrm{soc}(R_R)$.*

The statement of Proposition 2 in [3] includes the hypothesis that (eR, Rf) is an i-pair. However this hypothesis is not used in the proof, so the result should be stated as follows:

Lemma 2 ([3], Proposition 2)**.** *Let R be a semiprimary ring and let E be a basic set of primitive idempotents in R. If $e \in E$ and eR is simple-gR-injective for every $g \in E$, then eR is injective.*

Proposition 1. *Suppose R is semiperfect with a basic set $\{e_1, e_2, \ldots, e_n\}$ of primitive idempotents. If R is right simple-injective and $\mathrm{soc}(R_R) \subseteq^{\mathrm{ess}} R_R$, there exists a (Nakayama) permutation σ of $\{1, 2, \ldots, n\}$ such that $(e_kR, Re_{\sigma k})$ is an i-pair for each $k = 1, 2, \ldots, n$.*

Proof. The proof is divided into separate claims.

Claim 1. R is right Kasch.

Proof. For each i choose a simple right ideal $K_i \subseteq e_iR$. It suffices to show that the K_i are a set of distinct representatives of the simple right R-modules. Since the e_i are basic, we show that $K_i \cong K_j$ implies $e_iR \cong e_jR$. But if $\sigma\colon K_i \to K_j$ is an isomorphism, then $\sigma = a\cdot, a \in R$, because R is right simple-injective. If $\sigma^{-1} = b\cdot$ and $K_i = kR$, then $bak = k$ and it follows that $a \notin J$. Since we may assume $a \in e_j R e_i$, it follows that $a\cdot\colon e_i R \to e_j R$ is an isomorphism, proving Claim 1.

Claim 2. $rl(I) = I$ for every right ideal I of R.

Proof. If $b \in rl(I)$, $b \notin I$, let M/I be a maximal submodule of $(bR + I)/I$. If $\delta\colon (bR + I)/M \to R_R$ is an embedding (by Claim 1), define $\gamma\colon (bR + I) \to R$ by $\gamma(x) = \delta(x + M)$. Then $\gamma = c\cdot$ for some $c \in R$. Thus $cI = \gamma(I) = 0$, so $cb = 0$ because $b \in rl(I)$. But $cb = \delta(b + M) \neq 0$, proving Claim 2.

In particular, R is left principally injective because $rl(a) = aR$ for all $a \in R$. For convenience, write $S_r = \operatorname{soc}(R_R)$ and $S_l = \operatorname{soc}({}_R R)$.

Claim 3. $S_r \subseteq^{\text{ess}} {}_R R$. In particular $S_l \subseteq S_r$.

Proof. Let $0 \neq b \in R$. As in Claim 2, let $\gamma\colon bR \to R_R$ have simple image. Then $\gamma = c\cdot$ for $c \in R$, so $cb = \gamma(b) \neq 0$, while $cbJ \subseteq \gamma(bR)J = 0$. Thus $0 \neq cb \in Rb \cap l(J)$. Since $l(J) = S_r$ (R/J is semisimple), this proves Claim 3.

Claim 4. If kR is simple, $k \in R$, then $Rk = lr(k)$ is simple. In particular $S_l = S_r$.

Proof. Let kR be simple, $k \in R$. If $0 \neq a \in Rk$, then $r(k) \subseteq r(a)$ so $r(k) = r(a)$ because $r(k)$ is maximal. Hence $k \in lr(k) = lr(a)$ and it suffices to show that $lr(a) \subseteq Ra$. But if $b \in lr(a)$, then $\gamma\colon aR \to R$ is well defined by $\gamma(ar) = br$. As aR is simple, $\gamma = c\cdot$, $c \in R$, so $b = \gamma(a) = ca \in Ra$, proving Claim 4.

Thus S_l is essential as a left ideal (by Claims 3 and 4) so, since R is semiperfect and left principally injective, we conclude that R is a left GPF-ring.

Claim 5. If Rk is simple, $k \in R$, then kR is simple.

Proof. If Rk is simple, then $kR \subseteq S_r$ by Claim 4, so let $kR \supseteq mR$, mR simple. Then $l(k) \subseteq l(m)$ so $l(k) = l(m)$ (as $l(k)$ is maximal). But then $kR = mR$ by Claim 2, proving Claim 5.

Claim 6. $\operatorname{soc}(Re)$ is simple for every primitive idempotent $e \in R$.

Proof. We have $S_l \subseteq^{\text{ess}} {}_R R$ by Claims 3 and 4, so let $Rk \subseteq \operatorname{soc}(Re)$, Rk is a simple left ideal. Then $r(k) \supseteq (1 - e)R + J$ using Claim 5. But $(1 - e)R + J$ is a maximal right ideal (e is local), whence $r(k) = (1 - e)R + J$. Thus, using Claim 4, $Rk = lr(k) = Re \cap l(J) = Re \cap S_r = \operatorname{soc}(Re)$. This proves Claim 6.

Now, since R is a left GPF-ring, it follows from Lemma 1 and Claim 6 that there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $\operatorname{soc}(e_k R) \cong e_{\sigma k} R / e_{\sigma k} J$ and $\operatorname{soc}(R e_{\sigma k}) \cong R e_k / J e_k$ for each k. Hence $(e_k R, R e_{\sigma k})$ is an i-pair for each $k = 1, 2, \ldots, n$. This proves Proposition 1. □

The next proposition is now an easy consequence of Lemma 2.

Proposition 2. *Let R be a semiprimary ring. Then R is right self-injective if and only if R is right simple-injective.*

Proof. It is routine to see that if R is right simple-injective, then eR is simple-gR-injective for all primitive idempotents e and g. Hence each eR is injective by Proposition 1 and Lemma 2, so R_R is injective. □

Recall that $S_r \subseteq^{\text{ess}} R_R$ whenever R is left perfect. Hence the next result extends a well known theorem of Osofsky [15] which states that a left perfect, left and right self-injective ring is quasi-Frobenius.

Proposition 3. *Suppose R is a left perfect, left and right simple-injective ring. Then R is a quasi-Frobenius ring.*

Proof. From the proof of Proposition 1, it follows that R is a left GPF-ring in which $rl(I) = I$ for every right ideal I of R and $\operatorname{soc}({}_RR) = \operatorname{soc}(R_R) \subseteq^{\text{ess}} {}_RR$. By symmetry, $lr(L) = L$ for every left ideal L of R. Now [11, Theorem 5.3] shows that every principal left or right R-module has finite uniform dimension. As R is right semiartinian (R is left perfect), it follows that R is right artinian [4, Proposition 5]. In particular, R is semiprimary. Thus Proposition 2 implies that R is left and right self-injective. Whence R is quasi-Frobenius. □

The original proof of Proposition 4 below was lengthy and used set theoretic techniques. We are in debt to Professor Kent Fuller who communicated the present proof to us.

Proposition 4. *Suppose R is a semiperfect, right simple-injective ring with $\operatorname{soc}(R_R) \subseteq^{\text{ess}} R_R$. Given $n \geq 2$, if $\operatorname{soc}_n(R)$ is countably generated as a left R-module, then J^{n-1}/J^n is finitely generated as a right R-module.*

Proof. By Claims 3 and 4 of Proposition 1, we have $S_r = S_l \subseteq^{\text{ess}} {}_RR$. Write $S = S_r = S_l$. As R/J is semisimple, $S = l(J) = r(J)$.

Claim. For all $n \geq 1$, $\operatorname{soc}_n(R_R) = \operatorname{soc}_n({}_RR) = l(J^n) = r(J^n)$.

Proof. Suppose $\operatorname{soc}_k(R_R) = \operatorname{soc}_k({}_RR) = l(J^k) = r(J^k)$. We have $\operatorname{soc}_{k+1}(R_R) \subseteq l(J^{k+1})$ because $\operatorname{soc}_{k+1}(R_R)/\operatorname{soc}_k(R)$ is right R-semisimple. On the other hand, if $aJ^{k+1} = 0$, then $aJ \subseteq \operatorname{soc}_k(R)$ so $[aR + \operatorname{soc}_k(R)]/\operatorname{soc}_k(R)$ is right R-semisimple (as R/J is semisimple). Hence $aR \subseteq \operatorname{soc}_{k+1}(R_R)$ and we have $\operatorname{soc}_{k+1}(R_R) = l(J^{k+1})$. Similarly, $\operatorname{soc}_{k+1}({}_RR) = r(J^{k+1})$. Finally, $l(J^{k+1}) = r(J^{k+1})$ follows easily from $l(J^k) = r(J^k)$. This proves the claim.

We now assert that if I is a right ideal of R, every R-homomorphism $\varphi\colon I \to R$ with semisimple image is given by left multiplication by an element of R. Indeed, since R is right finite dimensional by Proposition 1, $\varphi(I) = \bigoplus_{i=1}^n S_i$ for simple right ideals S_i of R. Let $\pi_i\colon \bigoplus_{i=1}^n S_i \to S_i$ be the projection for each i, and write $\varphi_i = \pi_i \circ \varphi$. Since R is right simple-injective, there exist $t_i \in R$ such that $\varphi_i(a) = t_i a$ for all $a \in I$. If $t = \sum_{i=1}^n t_i$, then $\varphi(a) = ta$ for all $a \in I$.

With this it is straightforward to verify that

$$\hom_R(J^{n-1}/J^n, R_R) \cong l_R(J^n)/l_R(J^{n-1})$$

for all $n \geq 2$. This implies that J^{n-1}/J^n is finitely generated as a right R-module. Suppose not. Then, since R has finitely many isomorphism classes of simple right modules, let $S^{(\mathbb{N})}$ be a direct summand of J^{n-1}/J^n where S is a simple right R-module and $S^{(\mathbb{N})}$ denotes a countable direct sum of copies of S. Now ${}_RT = \hom_R(S, R_R)$ is a simple left R-module because R is right simple-injective and (by Proposition 1) right Kasch. Thus

$$T^{\mathbb{N}} = \hom_R(S, R)^{\mathbb{N}} \cong \hom_R(S^{(\mathbb{N})}, R) \hookrightarrow \operatorname{soc}_n(R)/\operatorname{soc}_{n-1}(R)$$

as a direct summand, where $T^{\mathbb{N}}$ is the direct product of countably many copies of T. But $T^{\mathbb{N}}$ has dimension $|T|^{|\mathbb{N}|} > |\mathbb{N}|$, according to a well known old theorem of Erdös and Kaplansky [6, Page 276], a contradiction. □

Observe that the proof of Proposition 4 actually yields the following: If R is a semiperfect, right simple-injective ring in which $\operatorname{soc}(R_R) \subseteq^{\text{ess}} R_R$ and $\operatorname{soc}_2(R)$ is generated on the left by χ elements, where χ is any ordinal number, then $(J/J^2)_R$ is generated by fewer than χ elements. For if this is not the case we can use the same argument to show that $\operatorname{soc}_2(R)/S$ contains a direct sum of $2^\chi > \chi$ simple modules, a contradiction. In particular, if $\operatorname{soc}_2(R)$ is generated on the left by ω elements (where ω is the first infinite ordinal), then $(J/J^2)_R$ is finitely generated.

Theorem 1. *Suppose R is a left perfect, right simple-injective ring. Then R is quasi-Frobenius if and only if* $\operatorname{soc}_2(R)$ *is countably generated as a left R-module.*

Proof. Proposition 4 and its proof show that $\operatorname{soc}_2(R_R) = \operatorname{soc}_2({}_RR)$, and that J/J^2 is finitely generated as a right R-module. Hence R is right artinian by Osofsky's theorem [15]. Then R is right self-injective by Proposition 2. Thus R is quasi-Frobenius. □

Theorem 2. *Suppose R is a left perfect right self-injective ring. Then R is quasi-Frobenius if and only if* $\operatorname{soc}_2(R)$ *is countably generated as a left R-module.*

Remarks. Similar arguments give the following results:

(1) If R is a semiperfect, right simple-injective ring with $\operatorname{soc}(R_R) \subseteq^{\text{ess}} R_R$, and if J/J^2 is countably generated as a left R-module, then $\operatorname{soc}_2(R)$ is finitely generated as a right R-module.

(2) If R is a left perfect, right self-injective ring, and if J/J^2 is countably generated as a left R-module, then $\operatorname{soc}_2(R)$ is finitely generated as a right R-module.

(3) If R is a left and right perfect, right self-injective ring, and if J/J^2 is countably generated as a left R-module, then R is quasi-Frobenius. (In fact, [8] shows that R is quasi-Frobenius if $\operatorname{soc}_2(R)$ is finitely generated as a right R-module, so (2) applies.)

Example 1. If R has zero right socle, then R is right simple-injective. Hence the ring of integers is an example of a commutative, noetherian, simple-injective ring which is not self-injective (or principally injective).

Example 2. We construct an example of a left perfect, left simple-injective ring S which is not right simple-injective (and hence not right self-injective). Let R be a left perfect ring which is not right perfect. Since R is left perfect, $\operatorname{soc}(R_R) \subseteq^{\text{ess}} R_R$ and $\operatorname{soc}_\lambda(R_R) = R$ for some ordinal λ. Since R is not right perfect (and hence not left semiartinian) $\operatorname{soc}_\alpha({}_RR) \neq R$ for any ordinal α. But R is a set, so we have $\operatorname{soc}_\beta({}_RR) = \operatorname{soc}_{\beta+1}({}_RR)$ for some ordinal β. Let $S = R/\operatorname{soc}_\beta({}_RR)$. Then S is a left perfect ring with $\operatorname{soc}({}_SS) = 0$, and so S is left simple-injective. But S cannot be right simple-injective by Proposition 3. In particular, S is not right self-injective.

A specific example is as follows: Let F be a field and let R be the ring of all lower triangular, countably infinite matrices over F with only finitely many off-diagonal entries. Let S be the F-subalgebra of R generated by 1 and $J(R)$. Then S is a left perfect, left simple-injective ring which is neither right perfect nor right simple-injective. Moreover, S is not left self-injective because it is not left finite dimensional.

Example 3 (Levy [13, page 115], see Hajarnavis-Norton [11, page 265])**.** Let $I = \{x | x \text{ is real and } 0 \leq x \leq 1\}$, let K be a field, and let R be the set of all formal sums of the form $\sum_{i \in I} a_i x^i$, where x is a commuting indeterminate over K, $a_i \in K$, and all but a finite number of the a_i are zero. Putting $x^k = 0$ for $k > 1$, R becomes a commutative ring by defining addition and multiplication in the usual way. It can be verified that R is a commutative, semiperfect, simple-injective, Kasch ring with essential socle, which satisfies $J^2 = J$. However, R is not self-injective.

Example 4 (Björk [5, page 70])**.** Let p be a prime number, let P be the field of p elements, and let $K = P(x)$ be the field of rational functions with coefficients in P. Then $K^p = \{w^p | w \in K\}$ is a subfield of K, $f \colon w \mapsto w^p$ is an isomorphism $K \to K^p$, and K^p is a p-dimensional vector space over K. If A is a left vector space over K with basis $\{e, x\}$, then A is a ring with multiplication defined by: $er = re = r$ for all $r \in A$, $x^2 = 0$ and $f(w)x = xw$ for all $w \in K = Ke$. The ring A is clearly left and right artinian. Moreover, $Ax = Kx$ is the only proper left ideal of A, so that every left ideal is an annihilator; in particular, A is right principally injective. But if $\{w_1, w_2, \ldots, w_p\}$ is a K-basis of K^p, then $w_i x A$, $i = 1, 2, \ldots, p$, are p distinct minimal right ideals of A so A is not quasi-Frobenius. In particular, A is not left or right simple-injective by Proposition 2.

Example 5 (Camillo [7, page 36])**.** Let R be the ring generated over the field of two elements by variables $x_1, x_2, \ldots$ where $x_i^3 = 0 = x_i x_j$ for all $i \neq j$, and $x_i^2 = x_j^2$ for all i and j. Then R is a commutative, local, semiprimary, principally injective ring, but R is not simple-injective by Proposition 2 because it is not artinian.

Conjecture. A left perfect, right simple-injective ring is right self-injective.

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