WILSON’S FUNCTIONAL EQUATION
FOR VECTOR AND MATRIX FUNCTIONS

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Abstract. We determine the general solution of the functional equation
\[ f(x+y) + f(x-y) = A(y)f(x) \quad (x,y \in G), \]
where \( G \) is a 2-divisible abelian group, \( f \) is a vector-valued function and \( A \) is a matrix-valued function. Using this result we solve the scalar equation
\[ f(x+y) + f(x-y) = g_1(x)h_1(y) + g_2(x)h_2(y) \quad (x,y \in G), \]
which contains as special cases, among others, the d’Alembert and Wilson equations and the parallelogram law.

1. Introduction

Let \( G \) be an abelian group divisible by 2 (that is, \( 2G = G \)) and \( K \) an algebraically closed commutative field of characteristic different from 2 and 3. Throughout this paper we denote by \( \mathcal{L}, \mathcal{M}, \mathcal{N} \) the sets of all solutions \( f: G \to K \) of the functional equations \( f(x+y) = f(x) + f(y) \), \( f(x+y) = f(x)f(y) \) and \( f(x+y) + f(x-y) = 2f(x) + 2f(y) \), respectively. Also, \( K_2 \) is the set of all \( 2 \times 2 \) matrices over \( K \), \( K^2 \) is the set of all \( 2 \times 1 \) matrices (column vectors) over \( K \), \( \mathcal{F} \) is the set of all vector-valued functions with linearly independent components, \( I \) is the unit matrix of \( K_2 \) and 0 (the zero of \( K \)) is also used for the zeros of \( G, K^2 \) and \( K_2 \). If \( a \in \mathcal{L} \), then \( a \) is called additive.

Clearly, if \( M \in K_2 \), \( f \in \mathcal{F} \) and \( Mf = 0 \), then \( M = 0 \).

In this paper we determine the general solutions \( f: G \to K^2 \), \( A: G \to K_2 \) of the functional equation
\[ f(x+y) + f(x-y) = A(y)f(x) \quad (x,y \in G), \]
which can be viewed as a vector analogue of Wilson’s functional equation (1.1) below (cf. also Eq. (3.4) of [4]).

The main result is given in Theorem 1. Using this result we solve in Theorem 2 the scalar equation
\[ f(x+y) + f(x-y) = g_1(x)h_1(y) + g_2(x)h_2(y) \quad (x,y \in G). \]

This last equation contains as special cases the Wilson equation
(1.1) \[ f(x+y) + f(x-y) = f(x)g(y). \]
introduced in [9], the equations  
(1.2) \[ f(x + y) + f(x - y) = 2f(x) + g(x)b(y), \]  
(1.3) \[ f(x + y) + f(x - y) = f(x)g(y) + g(x)f(y), \]  
(1.4) \[ f(x + y) + f(x - y) = f(x)f(y) + g(x)g(y), \]  
which were solved in [1], [3], and the parallelogram law  
(1.5) \[ f(x + y) + f(x - y) = 2f(x) + 2f(y), \]  
which characterizes the diagonalization of biadditive functions ([1], [2]). More general equations were solved e.g. in [4], but, as remarked in [3], to extract the solutions of (1.1)–(1.5) from [4] is not simple.

2. The results

**Theorem 1.** If \( A: G \to K_2, f: G \to K^2 \) with \( f \in \mathcal{F} \) is a solution of the functional equation  
(2.1) \[ f(x + y) + f(x - y) = A(y)f(x) \quad (x, y \in G), \]  
then  
(2.2) \[ \begin{cases} \ A(y) = C[E(y) + E(-y)]C^{-1}, \\ f(x) = C[E(x)\alpha + E(-x)\beta], \end{cases} \]  
where \( C \in K_2 \) (det \( C \neq 0 \)), \( \alpha, \beta \in K^2 \) and \( E \) has one of the forms  
(2.3) \[ E(x) = \chi(x) \begin{bmatrix} 1 & \phi(x) \\ 0 & 1 \end{bmatrix}, \]  
(2.4) \[ \begin{bmatrix} 1 & n(x) + \phi(x) \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \phi(x) + 1 & c(\phi(x))^3 + 3c\phi(x)^2 + \phi_1(x) \\ 0 & \phi(x) + 1 \end{bmatrix}, \]  
(2.5) \[ \begin{bmatrix} \chi_1(x) - [\chi_1(0) - 1][\phi_1(x) + 1] \\ 0 \chi_2(x) - [\chi_2(0) - 1][\phi_2(x) + 1] \end{bmatrix}, \]  
with \( c \in K, \phi, \phi_1, \phi_2 \in \mathcal{L}, \chi, \chi_1, \chi_2 \in \mathcal{M}, \chi(x) \neq 0, \) and \( n \in \mathcal{N}. \)  

Conversely, any \( A, f \) as described above satisfy (2.1).

**Note.** If \( f \notin \mathcal{F}, \) then (2.1) reduces to a system of two Wilson equations and the solutions are obtained by application of Theorem (2.2) of [5].

**Proof.** Setting \( y = u + v \) and \( y = u - v \) in (2.1), adding the resulting equations and using (2.1) again we obtain  
(2.4) \[ A(v)A(u) = [A(u + v) + A(u - v)]f(x). \]  
Since \( f \in \mathcal{F} \) this leads to the matrix d’Alembert equation  
(2.4) \[ A(v)A(u) = A(u + v) + A(u - v). \]  
Replacing \( y \) by \( -y \) in (2.1) we see that \( A(y)f(x) = A(-y)f(x) \) and so \( A(y) = A(-y). \) Hence, interchanging \( u, v \) in (2.4), we have \( A(u)A(v) = A(v)A(u). \) Since \( K \) is algebraically closed there exists an invertible matrix \( C \in K_2 \) such that the matrix function \( B(x) := C^{-1}A(x)C \) has the form  
(2.4) \[ B(x) = \begin{bmatrix} b_1(x) & b_0(x) \\ 0 & b_2(x) \end{bmatrix}, \]  
with \( b_1(x) \equiv b_2(x) \) or \( b_0(x) \equiv 0 \) (see [8], [4]).
Let \( g(x) := C^{-1}f(x) \). Substituting
\[
(2.5) \quad A(x) = CB(x)C^{-1} \quad \text{and} \quad f(x) = Cg(x)
\]
into (2.1) we get
\[
(2.6) \quad g(x + y) + g(x - y) = B(y)g(x),
\]
that is, with \( g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \),
\[
(2.6a) \quad g_1(x + y) + g_1(x - y) = g_1(x)b_1(y) + g_2(x)b_0(y),
\]
\[
(2.6b) \quad g_2(x + y) + g_2(x - y) = g_2(x)b_2(y).
\]
Substituting also the first of (2.5) into (2.4) we have
\[
(2.7) \quad B(u + v) + B(u - v) = B(u)B(v),
\]
that is,
\[
(2.7a) \quad b_i(u + v) + b_i(u - v) = b_i(u)b_i(v), \quad i = 1, 2,
\]
\[
(2.7b) \quad b_0(u + v) + b_0(u - v) = b_1(u)b_0(v) + b_0(u)b_2(v).
\]
We distinguish three cases.

Case 1. \( b_1(x) \equiv b_2(x) \neq 2 \). Then \( b_1(x_0) = b_2(x_0) \neq 2 \) for some \( x_0 \in G \). Setting \( y = 0 \) in (2.6) gives \( B(0) = 2I \) (since \( g \in \mathcal{F} \)) and from (2.7) it follows that \( B(2x) + 2I = B(x)^2 \). So the matrix \( B(2x_0) - 2I = B(x_0)^2 - 4I \) is invertible. This and all subsequent \( 2 \times 2 \) matrices are upper triangular with equal diagonal elements and hence commute.

Now the solutions of the matrix equation (2.7) can be written in the form \( B(x) = \Xi(x) + \Xi(-x) \) where \( \Xi(x) = \chi(x) \begin{pmatrix} 1 & \phi(x) \\ 0 & 1 \end{pmatrix} \) with \( \phi \in \mathcal{L}, \chi \in \mathcal{M} \) and \( \Xi(x + y) = \Xi(x)\Xi(y) \) (see, e.g., the proof of Lemma 1 of [7]).

In order to solve the equation (2.6) we follow the method used in the proof of Theorem 2.2 of [5]. In (2.6) we replace \( x \) by \( -x \), we subtract the resulting equation from (2.6) and, setting \( h(x) = g(x) - g(-x) \), we obtain
\[
h(x + y) + h(x - y) = B(y)\bar{h}(x)
\]
with \( h(0) = 0, h(-x) = -h(x) \) and \( h(2x) = B(x)h(x) \). Therefore
\[
[2B(x + y) - B(x)B(y)]h(x) = 2B(x + y)h(x) - B(y)\bar{h}(2x)
\]
\[
= 2[h(2x + y) - h(y)] - [h(2x + y) + h(2x - y)]
\]
\[
= h(y + 2x) + h(y - 2x) - 2h(y) = B(2x)h(y) - 2h(y).
\]
Since \( 2B(x + y) - B(x)B(y) = [\Xi(x) - \Xi(-x)][\Xi(y) - \Xi(-y)] \) we have
\[
[B(2x) - 2I]h(y) = [\Xi(x) - \Xi(-x)][\Xi(y) - \Xi(-y)]\bar{h}(x).
\]
Setting \( x = x_0 \) we obtain
\[
g(y) - g(-y) = [\Xi(y) - \Xi(-y)]\gamma
\]
where \( \gamma \in K^2 \). Adding this to \( g(y) + g(-y) = [\Xi(y) + \Xi(-y)]g(0) \) (which follows from (2.6) with \( x = 0 \)) we obtain
\[
g(x) = \Xi(x)\alpha + \Xi(-x)\beta,
\]
where \( \alpha, \beta \in K^2 \). So by (2.5) we obtain (2.2) with the first form in (2.3).
Case 2. \( b_1(x) \equiv b_2(x) \equiv 2 \). Then (2.7b) shows that \( b_0 \in \mathcal{N} \) and from (2.6b) (a Jensen equation) we have \( g_2(x) = \phi_2(x) + \lambda \) with \( \lambda \in K \) and \( \phi_2 \in \mathcal{L} \). Now (2.6a) becomes

\[
g_1(x + y) + g_1(x - y) = 2g_1(x) + [\phi_2(x) + \lambda]b_0(y). \tag{2.8}
\]

If \( \phi_2(x) \equiv 0 \), then interchanging \( x, y \) in (2.8) and subtracting we find

\[
g_1(x - y) - g_1(y - x) = [2g_1(x) - \lambda b_0(x)] - [2g_1(y) - \lambda b_0(y)],
\]

a Pexider equation which gives \( 2g_1(x) - \lambda b_0(x) = 2\phi_0(x) + 2\mu \) (\( \mu \in K, \phi_0 \in \mathcal{L} \)). Hence

\[
g(x) = \left\| \frac{1}{2} \lambda b_0(x) + \phi_0(x) + \mu \right\|, \quad B(y) = \left\| \frac{2}{0} b_0(y) \right\|.
\]

Using (2.5) we obtain (2.2) with the second form in (2.3).

If \( \phi_2(x) \not\equiv 0 \), then applying Lemma 4 of [1] to (2.8) (here we need the condition \( \text{char } K \neq 2, 3 \)) we obtain

\[
g_1(x) = c\phi_2(x)^3 + 3c\phi_2(x)^2 + \phi_0(x) + \mu, \quad b_0(y) = 6c\phi_2(y)^2,
\]

where \( c, \mu \in K \) and \( \phi_0 \in \mathcal{L} \). So by (2.5) we obtain (2.2) with the third form in (2.3).

Case 3. \( b_0(x) \equiv 0 \). Then applying Theorem 2.2 of [5] to (2.6a) and (2.6b) we see that \( g(x), B(y) \) have one of the forms

\[
g(x) = \begin{cases} \frac{1}{2} \lambda b_0(x) + \phi_0(x) + \mu & \text{if } \theta, \eta \in \mathcal{M}, a_0, \phi_0 \in \mathcal{L} \text{ and } \kappa, \lambda, \mu, \nu \in K, \\ \frac{1}{2} \lambda b_0(x) + \phi_0(x) + \mu & \text{if } \theta, \eta \in \mathcal{M}, a_0, \phi_0 \in \mathcal{L} \text{ and } \kappa, \lambda, \mu, \nu \in K, \end{cases}
\]

\[
B(y) = \begin{cases} \left\| \theta(y) + \theta(-y) \right\| & \text{if } \theta, \eta \in \mathcal{M}, a_0, \phi_0 \in \mathcal{L} \text{ and } \kappa, \lambda, \mu, \nu \in K, \\ \left\| \theta(y) + \theta(-y) \right\| & \text{if } \theta, \eta \in \mathcal{M}, a_0, \phi_0 \in \mathcal{L} \text{ and } \kappa, \lambda, \mu, \nu \in K, \end{cases}
\]

where \( \theta, \eta \in \mathcal{M}, a_0, \phi_0 \in \mathcal{L} \) and \( \kappa, \lambda, \mu, \nu \in K \). So by (2.5) we obtain (2.2) with the last form in (2.3).

To prove the converse observe that the substitution of (2.2) into (2.1) gives

\[
\begin{align*}
[E(x + y) + E(x - y) - [E(y) + E(-y)]E(x)]\alpha \\
+ [E(-x + y) + E(-x - y) - [E(y) + E(-y)]E(-x)]\beta &= 0.
\end{align*}
\]

On the other hand, it is easy to verify that the functions \( E \) in (2.3) satisfy the functional equation

\[
E(x + y) + E(x - y) = [E(y) + E(-y)]E(x) \quad (x, y \in G)
\]

and this completes the proof of Theorem 1. \( \square \)

In the following theorem the prime indicates transposition.

**Theorem 2.** If \( f : G \to K, g, h : G \to K^2 \) is a solution of the functional equation

\[
f(x + y) + f(x - y) = h'(y)g(x) \quad (x, y \in G), \tag{2.10}
\]

\[
t : G \to K^2 \text{ is a solution of the functional equation}
\]

\[
t(x + y) + t(x - y) = (G(y) + G(-y))t(x) \quad (x, y \in G),
\]

and hence \( t \) satisfies (2.2).

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then $f$ has the form
\begin{equation}
\tag{2.11}
 f(x) = \lambda[E(x)\alpha + E(-x)\beta],
\end{equation}
where $\lambda, \alpha, \beta \in K^2$ and $E$ is given by (2.3).

Conversely, for any $f$ as described above there exist $g, h$ which together with $f$
constitute a solution of (2.10).

Notes. 1. Once $f$ is known, it is easy to determine the complete list for the vector-valued functions $g, h$.

2. The right-hand side of (2.10), as a scalar product, can also be written
\[ g'(x)h(y) \]

Proof. If $g \not\in \mathcal{F}$ or $h \not\in \mathcal{F}$, then (2.10) reduces to a Wilson equation. Applying
Theorem 2.2 of [5] to this equation we see that the solutions are contained in
(2.11). So in the following we assume that $g, h \in \mathcal{F}$.

Setting $y = u + v$ and $y = u - v$ in (2.10), adding the resulting equations and
using again (2.10) we find
\[
 h'(v)(g(x + u) + g(x - u)) = [h'(u + v) + h'(u - v)]g(x).
\]

Since $h \in \mathcal{F}$, using Lemma 14.1 of [2] we obtain
\[
 g(x + u) + g(x - u) = A(u)g(x),
\]
with $A : G \to K_2$. Since $g \in \mathcal{F}$, applying Theorem 1 to this equation we obtain
\[
 g(x) = C[E(x)\alpha + E(-x)\beta],
\]
where, $\alpha, \beta \in K^2$, $C \in K_2$ and $E$ is given by (2.3).

Now from (2.10) with $y = 0$ we have $f(x) = \frac{1}{2}h'(0)g(x)$ which leads to (2.11).

The converse is verified by substitution of (2.11) into (2.10) and using (2.9). \hfill \square

Remark. The following result generalizes Theorem 1 and is proved in the same way. Here $S$ is an arbitrary set.

**Theorem 3.** If $A : G \to K_2$, $f : G \times S \to K^2$ with $f \in \mathcal{F}$ is a solution of the functional equation
\begin{equation}
\tag{2.12}
 f(x + y, s) + f(x - y, s) = A(y)f(x, s) \quad (x, y \in G, s \in S),
\end{equation}
then
\begin{equation}
\tag{2.13}
\begin{cases}
 A(y) = C[E(y, s_0) + E(-y, s_0)]C^{-1}, \\
 f(x, s) = C[E(x, s)\alpha(s) + E(-x, s)\beta(s)],
\end{cases}
\end{equation}
where $s_0 \in S$, $C \in K_2$ (det $C \neq 0$), $\alpha, \beta : S \to K^2$ are arbitrary functions and $E(x, s)$ has one of the forms
\begin{equation}
\tag{2.14}
 E(x, s) = \chi(x) \begin{vmatrix}
 1 & \phi(x) \\
 0 & 1 \\
\end{vmatrix},
\end{equation}
with $c \in K, \phi \in \mathcal{L}, \chi, \chi_1, \chi_2 \in \mathcal{M}, \chi(x) \neq 0, n \in \mathcal{N}$ and $\phi_1, \phi_2 : G \times S \to K$
additive in the first variable.

Conversely, any $A, f$ as described above satisfy (2.12).
(Clearly, for $E$ given by (2.14), the sum $E(y, s) + E(-y, s)$ is independent of the second variable, that is, $E(y, s) + E(-y, s) = E(y, t) + E(-y, t)$ for all $y \in G$ and $s, t \in S$. So $A$ in the first of (2.13) is independent of the choice of $s_0$.)

Theorem 3 can be used, e.g., in the solution of the functional equation

$$f(x + t, y - t) + f(x - t, y + t) = g_1(x, y)h_1(t) + g_2(x, y)h_2(t)$$

which generalizes (2.10) (and (3.2) of [6]).

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**References**


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