

## RUDIN'S ORTHOGONALITY PROBLEM AND THE NEVANLINNA COUNTING FUNCTION

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**ABSTRACT.** Let  $\phi$  be a holomorphic function taking the open unit disk  $U$  into itself. We show that the set of nonnegative powers of  $\phi$  is orthogonal in  $L^2(\partial U)$  if and only if the Nevanlinna counting function of  $\phi$ ,  $N_\phi$ , is essentially radial. As a corollary, we obtain that the orthogonality of  $\{\phi^n : n = 0, 1, 2, \dots\}$  for a univalent  $\phi$  implies  $\phi(z) = \alpha z$  for some constant  $\alpha$ . We also show that if  $\{\phi^n : n = 0, 1, 2, \dots\}$  is orthogonal, then the closure of  $\phi(U)$  must be a disk.

### 1. INTRODUCTION

Observe that if  $f$  is an inner function on the open unit disk  $U$  satisfying  $f(0) = 0$ , then  $\{f^n : n = 0, 1, 2, \dots\}$  is orthogonal in  $L^2(\partial U)$ . The following problem was posed by Walter Rudin in 1988 (at an MSRI conference):

*Question 1* (Rudin). If  $f$  is a bounded analytic function on the open unit disk  $U$  such that  $\{f^n : n = 0, 1, 2, \dots\}$  is orthogonal in  $L^2(\partial U)$ , must  $f$  be a constant multiple of an inner function?

Note that if the  $H^\infty$  function  $f$  satisfies the orthogonality condition of Rudin's question, then  $f(0) = 0$ . Also note that any constant multiple of  $f$  will satisfy the condition; thus, we may restrict our attention to maps  $f$  taking  $U$  into itself. For the remainder of this paper,  $\phi$  will denote an analytic map taking  $U$  into  $U$ .

Suppose that  $\phi$  is inner and fixes the origin; then according to Littlewood's Inequality (discussed in the next section), the Nevanlinna counting function of  $\phi$ ,  $N_\phi$ , is given by  $N_\phi(w) = \log(1/|w|)$  for all  $w$  outside of some subset of  $U$  having logarithmic capacity 0. Thus,  $N_\phi$  is essentially a radial function when  $\phi$  is an inner function vanishing at zero. In this paper, we prove that Rudin's orthogonality problem is equivalent to the following.

*Question 2.* Suppose that for almost every  $r \in [0, 1]$ ,  $\theta \mapsto N_\phi(re^{i\theta})$  is essentially constant; must  $\phi$  be a constant multiple of an inner function?

That the preceding question is equivalent to Rudin's Question 1 follows from our Main Theorem.

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**Main Theorem.** *Let  $\phi$  be a self-map of  $U$  fixing 0. The set  $\{\phi^n : n = 0, 1, 2, \dots\}$  is orthogonal in  $L^2(\partial U)$  if and only if there is a function  $g : [0, 1] \rightarrow [0, \infty)$  such that for almost every  $r \in [0, 1]$ ,  $N_\phi(re^{i\theta}) = g(r)$  for almost every  $\theta \in [0, 2\pi]$ .*

We obtain as corollaries of the Main Theorem two results that suggest that the answer to Rudin's question is "yes". First we show the answer is "yes" if  $\phi$  is assumed to be univalent (see Corollary 1). Then we show that if  $\phi$  satisfies the orthogonality hypothesis, the closure of  $\phi(U)$  must be a disk (see Corollary 2). Cima, Korenblum, and Stessin [2] have also obtained evidence that Rudin's question has an affirmative answer: they show that if  $f$  is in the disk algebra with boundary function in  $\text{Lip}_\alpha$  for some  $\alpha > 1/2$  and if the set of nonnegative powers of  $f$  is orthogonal in  $L^2(\partial U)$ , then  $f$  must be a multiple of an inner function. They also obtain a result similar to our Corollary 2 (see [2, Corollary 1]).

## 2. BACKGROUND

A bounded analytic function on  $U$  has finite radial limit at almost every point on the boundary of  $U$  (see, e.g., [3, Theorem 1.3]). If the radial limit function has modulus 1 almost everywhere, then the function is said to be *inner*.

Let  $f$  and  $g$  be bounded and analytic on  $U$ . The inner product of  $f$  and  $g$  in  $L^2(\partial U)$ , which we will denote  $\langle f, g \rangle$ , may be computed in several ways:

$$\begin{aligned} \langle f, g \rangle &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta \\ &= \sum_{k=0}^{\infty} \hat{f}(k) \overline{\hat{g}(k)} \\ &= \frac{1}{\pi} \int_U f'(z) \overline{g'(z)} \log(1/|z|^2) dA(z) + f(0) \overline{g(0)}. \end{aligned}$$

In the first representation of the inner product,  $f(e^{i\theta})$  and  $g(e^{i\theta})$  denote, respectively, the radial limits of  $f$  and  $g$  at  $e^{i\theta}$ . In the second,  $\hat{f}(n)$  and  $\hat{g}(n)$  denote, respectively, the  $n$ -th Taylor coefficients of  $f$  and  $g$  in the expansion about the origin. The third representation is the Littlewood-Paley formula, which may be proved using power series ( $dA$  represents area measure).

The first step in our proof of the Main Theorem entails the use of the third representation of the inner product followed by a change of variables. The change of variables introduces the Nevanlinna counting function. For a holomorphic map  $\phi$  taking  $U$  into itself, the Nevanlinna counting function of  $\phi$ ,  $N_\phi$ , is defined on  $U \setminus \{\phi(0)\}$  by

$$N_\phi(w) = \sum_{\phi(z)=w} \log(1/|z|),$$

where multiplicities are counted and  $N_\phi(w)$  is taken to be zero if  $w$  is not in the range of  $\phi$ . The Nevanlinna counting function of  $\phi$  is lower semicontinuous and satisfies the sub-mean-value property ([5, Section 2]). Some other important properties of  $N_\phi$  are described in the following result due to Littlewood ([6], see also [8, 4.2]).

**Littlewood's Inequality.** *For each  $w \in U \setminus \{\phi(0)\}$ ,*

$$(\dagger) \quad N_\phi(w) \leq \log \left| \frac{1 - \overline{\phi(0)}w}{\phi(0) - w} \right|;$$

moreover, the following are equivalent:

- equality occurs in (†) at some  $w$  in  $U$ ;
- equality occurs in (†) at all points in  $U$  outside a subset of  $U$  having zero logarithmic capacity;
- $\phi$  is an inner function.

The preceding result shows that inner functions that vanish at 0 have Nevanlinna counting functions that are essentially radial. As we pointed out in the introduction, our work shows that Rudin's orthogonality problem is equivalent to "If  $N_\phi$  is radial, must  $\phi$  be inner?"

### 3. RESULTS

The following lemma is an easy consequence of the Müntz-Szasz Theorem (see, e.g., [7, p. 336]; see also the note added in proof at the end of this paper).

**Müntz-Szasz Lemma.** *Suppose that  $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$  and  $\sum_{k=1}^\infty 1/\lambda_k$  diverges. Then the set of all finite linear combinations of the functions defined on  $[0, 1]$  by*

$$t \mapsto t^{\lambda_1}, t \mapsto t^{\lambda_2}, t \mapsto t^{\lambda_3}, \dots$$

*is dense in  $L^2([0, 1])$ .*

*Proof.* By the Müntz-Szasz Theorem, the set of all finite linear combinations of

$$t \mapsto 1, t \mapsto t^{\lambda_1}, t \mapsto t^{\lambda_2}, t \mapsto t^{\lambda_3}, \dots$$

is dense in  $C([0, 1])$  (where, as usual,  $C([0, 1])$  denotes the set of continuous, complex-valued functions on  $[0, 1]$  with the supremum norm). It follows that any continuous function on  $[0, 1]$  that vanishes at 0 may be uniformly approximated by functions in the linear span of  $\{t \mapsto t^{\lambda_1}, t \mapsto t^{\lambda_2}, t \mapsto t^{\lambda_3}, \dots\}$ . However, the set of continuous functions vanishing at 0 is dense in  $L^2([0, 1])$ .  $\square$

**Main Theorem.** *Let  $\phi$  be a self-map of  $U$  fixing 0. The following are equivalent:*

- (a) *the set  $\{\phi^n : n = 0, 1, 2, \dots\}$  is orthogonal in  $L^2(\partial U)$ ;*
- (b) *there is a function  $g : [0, 1] \rightarrow [0, \infty)$  such that for almost every  $r \in [0, 1]$ ,  $N_\phi(re^{i\theta}) = g(r)$  for almost every  $\theta \in [0, 2\pi]$ .*

*Proof.* Suppose (b) holds. Let  $n > m$  be nonnegative integers and let  $d\lambda = dA/\pi$ . We have

$$\begin{aligned} \langle \phi^n, \phi^m \rangle &= nm \int_U \phi(z)^{n-1} \overline{\phi(z)^{m-1}} |\phi'(z)|^2 \log(1/|z|^2) d\lambda(z) \\ &= 2nm \int_U z^{n-1} \overline{z^{m-1}} N_\phi(z) d\lambda(z) \\ &= 2nm \int_0^1 r^{n+m-1} \left[ \int_0^{2\pi} e^{i(n-m)\theta} N_\phi(re^{i\theta}) d\theta \right] dr \\ &= 2nm \int_0^1 r^{n+m-1} g(r) \left[ \int_0^{2\pi} e^{i(n-m)\theta} d\theta \right] dr \\ &= 0. \end{aligned}$$

The change of variables yielding the second equality is justified in, e.g., [8, 4.3].

Conversely, suppose  $\{\phi^n : n = 0, 1, 2, \dots\}$  is orthogonal. Let  $k$  be an arbitrary positive integer. For each integer  $n$  larger than  $k$ , we have

$$\begin{aligned} 0 &= \langle \phi^n, \phi^{n-k} \rangle \\ &= n(n-k) \int_U \phi(z)^{n-1} \overline{\phi(z)^{n-k-1}} |\phi'(z)|^2 \log(1/|z|^2) d\lambda(z) \\ &= 2n(n-k) \int_U z^{n-1} \overline{z^{n-k-1}} N_\phi(z) d\lambda(z) \\ &= 2n(n-k) \int_0^1 r^{2n-k-1} \left[ \int_0^{2\pi} e^{ik\theta} N_\phi(re^{i\theta}) d\theta \right] dr. \end{aligned}$$

Thus the function  $f_k(r) := \int_0^{2\pi} e^{-ik\theta} N_\phi(re^{i\theta}) d\theta$  is orthogonal in  $L^2([0, 1])$  to the linear span of the set  $\{r \mapsto r^{2n-k-1} : n > k\}$  (Littlewood's inequality shows that  $|f_k(r)| \leq 2\pi \log(1/r)$ , so  $f_k$  is in  $L^2([0, 1])$ ). By the Muntz-Szasz Lemma, the linear span of  $\{r \mapsto r^{2n-k-1} : n > k\}$  is dense in  $L^2([0, 1])$ ; hence,  $f_k(r) = 0$  for almost every  $r$  in  $[0, 1]$ . Taking complex conjugates, we also have that  $f_{-k}$  vanishes at almost every point of  $[0, 1]$ . Since  $k$  is an arbitrary positive integer, we may conclude that there is a set  $E$  of full measure in  $[0, 1]$  such that for each  $r \in E$ ,

$$\int_0^{2\pi} e^{ij\theta} N_\phi(re^{i\theta}) d\theta = 0 \quad \text{for all } j \neq 0.$$

Thus for fixed  $r \in E$ , we see that  $\theta \mapsto N_\phi(re^{i\theta})$  is essentially constant, equaling, say,  $g(r)$ . Set  $g(r) = 0$  for  $r \in [0, 1] \setminus E$ , and the proof is complete.  $\square$

The following two corollaries provide some evidence that the answer to Rudin's orthogonality question is "yes".

**Corollary 1.** *Suppose that  $\phi$  is a univalent self-map of  $U$  the set of whose non-negative powers is orthogonal in  $L^2(\partial U)$ . Then there is a constant  $\alpha$  such that  $\phi(z) = \alpha z$ .*

*Proof.* We know  $\phi(0) = 0$  because  $\phi^1$  is orthogonal to  $\phi^0$ . Let  $D$  be a disk centered at 0 and contained in  $\phi(U)$ . On  $D \setminus \{0\}$ ,

$$N_\phi(w) = \log(1/|\phi^{-1}(w)|),$$

so that  $N_\phi$  is continuous (in fact, harmonic) on  $D \setminus \{0\}$ . By continuity and the Main Theorem, we see that  $N_\phi$  is a radial function on  $D \setminus \{0\}$ . Because  $N_\phi$  is radial and harmonic on  $D \setminus \{0\}$ , there are constants  $b$  and  $c$  such that on  $D \setminus \{0\}$

$$N_\phi(w) = b \log |1/w| + c$$

(see, e.g., [1, Lemma 3.10]). Exponentiating both sides of the equation above and composing with  $\phi$ , we have for each  $z \in \phi^{-1}(D \setminus \{0\})$

$$\frac{1}{|z|} = \left| \frac{1}{\phi(z)} \right|^b e^c.$$

Observe that

$$\lim_{z \rightarrow 0} |\phi(z)|^{b-1} = e^c / |\phi'(0)|;$$

and because  $\phi(0) = 0$ , it follows that  $b = 1$ . Thus

$$\left| \frac{\phi(z)}{z} \right| = e^c$$

for  $z$  near 0, and thus  $\phi(z)/z$  must be constant near zero, hence constant on all of  $U$ . □

**Corollary 2.** *Suppose that  $\phi$  is a self-map of  $U$  the set of whose nonnegative powers is orthogonal in  $L^2(\partial U)$ . Then the closure of  $\phi(U)$  is a disk.*

*Proof.* If  $\phi$  is constant, then  $\phi \equiv 0$  and  $\phi(U) = \{0\}$ . Assume that  $\phi$  is nonconstant. We know that there is a set  $E \subset [0, 1]$  of measure 1 such that for each  $r \in E$ ,  $N_\phi(re^{i\theta})$  is essentially constant in  $\theta$ . Let  $r \in E$ . If the range of  $\phi$  contains a point on the circle  $C_r := \{re^{i\theta} : \theta \in [0, 2\pi]\}$  (hence an entire interval by the open mapping theorem), then the range of  $\phi$  may omit only a  $\theta$ -measure-zero subset of  $C_r$  because

- (1)  $N_\phi$  is essentially constant on  $C_r$ , and
- (2)  $N_\phi(w) = 0$  if and only if  $w$  is not in the range of  $\phi$ .

Thus if the range hits  $C_r$ , the closure of the range must contain  $C_r$ . Hence the closure of the range contains  $C_r$  for  $r \in E \cap [0, \|\phi\|_\infty)$ . Because  $E$  has full measure in  $[0, 1]$ , the closure of the range must be the closed disk of radius  $\|\phi\|_\infty$ . □

Using different methods, Stephen Fisher has obtained a result stronger than the preceding corollary: he has shown that if the set of nonnegative powers of  $\phi$  is orthogonal, then  $\phi(U)$  must be a disk except possibly for a set of logarithmic capacity zero. We conclude this note with a proof of Fisher's result based on the Main Theorem.

**Theorem (Fisher).** *Suppose that  $\phi$  is a self-map of  $U$  the set of whose nonnegative powers is orthogonal in  $L^2(\partial U)$ ; suppose also that  $\|\phi\|_\infty = 1$ . Then there is a subset  $E$  of  $U$  having logarithmic capacity zero such that  $\phi(U) = U \setminus E$ .*

*Proof.* There is a subset  $E$  of  $U$  having capacity zero such that on  $U \setminus E$  the function  $N_\phi$  is continuous (see, e.g., p. 327 of [4]). That  $N_\phi$  is radial on  $U \setminus E$  is an easy consequence of our Main Theorem—thus, if  $w_1$  and  $w_2$  are points of continuity of  $N_\phi$  with  $|w_1| = |w_2|$ , we have  $N_\phi(w_1) = N_\phi(w_2)$ .

Suppose that  $\phi(U)$  omits a subset  $H$  of  $U$  having positive capacity. Let  $w_0$  be a point of continuity of  $N_\phi$  contained in  $H$ . Because  $w_0$  is not in the range of  $\phi$ , we have  $N_\phi(w_0) = 0$ . Because  $w_0$  is a point of continuity of  $N_\phi$ , we see that  $N_\phi(w) = 0$  at each point  $w$  of continuity with  $|w| = |w_0|$ . Now observe that there must be a point on the circle of radius  $|w_0|$  that is in  $\phi(U)$  ( $\phi(0) = 0$  by the orthogonality condition, and, by hypothesis,  $\|\phi\|_\infty = 1$ ). Because  $\phi$  is an open mapping,  $\phi(U)$  contains a nonempty open arc  $J$  of points on the circle  $\{z : |z| = |w_0|\}$ . This arc  $J$  has positive capacity and hence contains a point of continuity of  $N_\phi$ ; at this point of continuity we must have  $N_\phi = 0$ , but this contradicts the fact that every point of  $J$  lies in  $\phi(U)$ . □

*Remark.* Because  $N_\phi$  is continuous off of a set of logarithmic capacity zero, we may rephrase Question 2 of the introduction as follows. Suppose that  $N_\phi$  agrees with a continuous, radial, subharmonic function off of a set of logarithmic capacity zero; must  $\phi$  be a constant multiple of an inner function?

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## ADDED IN PROOF

Alec Matheson has pointed out to the author that a complete characterization of those subsets of  $\{t \mapsto t^p : p > -1/2\}$  that span a dense subset of  $L^2([0, 1])$  may be found in *Interpolation and Approximation* by P. J. Davis (Dover, New York, 1975, p. 270). Davis attributes the characterization to Müntz.

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