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ANALYSIS OF THE WU METRIC II: THE CASE OF NON-CONVEX THULLEN DOMAINS

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Abstract. We present an explicit description of the Wu invariant metric on the non-convex Thullen domains. We show that the Wu invariant Hermitian metric, which in general behaves as nicely as the Kobayashi metric under holomorphic mappings, enjoys better regularity in this case. Furthermore, we show that the holomorphic curvature of the Wu metric is bounded from above everywhere by $-1/2$. This leads the Wu metric to be a natural solution to a conjecture of Kobayashi in the case of non-convex Thullen domains.

1. Introduction

In the study of complex geometry and several complex variables, three different types of invariant metrics play an important role. They are the Kobayashi metric [16], the Poincaré-Bergman metric [4] and the Einstein-Kähler metric [7]. The Kobayashi metric behaves nicely under holomorphic mapping (it has the distance decreasing property), and is therefore a powerful tool in the study of complex analysis. However, in general it is only a Finsler metric and does not have nice regularity, which limits its application. On the other hand, the Poincaré-Bergman metric and the Einstein-Kähler metric are Kähler metrics and have nice curvature properties, but they are not well behaved under holomorphic mapping. Ideally, one would hope to mediate between these two approaches and find an invariant metric that is well behaved under holomorphic mapping and at the same time has nice geometrical properties.

In [24], Wu introduced a new invariant *Hermitian* metric. Throughout this paper we call this metric the *Wu metric*. It is the first known intrinsic invariant Hermitian metric that possesses the distance decreasing property up to a fixed constant factor. As mentioned above, the distance decreasing property plays a major role in applying the Kobayashi metric in the study of analytic function theory. But this property is not shared by the afore-mentioned invariant Kählerian metrics. In [8] we studied the behavior of the Wu metric on the convex Thullen domain defined by

$$E_\lambda = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^{2\lambda} < 1\}$$

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where λ is a real number not smaller than $1/2$. We proved that the Wu metric is real analytic everywhere except along a thin set and C^1 smooth overall, and enjoys nice curvature properties as compared to the other Kählerian invariant metrics. (Namely, the Wu metric has negative holomorphic curvature everywhere in E_λ, $\lambda \geq 1/2$. Moreover, when $\lambda > 1$, the holomorphic curvature is identically -2 in a large open subset of E_λ.)

In this paper, we continue the study of the Wu metric on E_λ with $0 < \lambda < 1/2$, the *non-convex* Thullen domains. (The Wu metric on this domain is complete since the Kobayashi metric is complete. See Hahn-Pflug [11].) We show that the Wu metric is real analytic except along a thin subset and that its holomorphic curvature is negative everywhere.

So far, most known examples of bounded domains that admit negatively curved complete metrics are convex or essentially convex ([2], [3], [9], [10], [14], [15] to cite only a few). Note that the domain E_λ, $0 < \lambda < 1/2$, cannot be biholomorphically mapped onto any bounded convex domain in $\mathbb{C}^2$, since its Kobayashi indicatrix is not convex ([18]). This adds more interest to our results because it provides the first example of an essentially non-convex domain possessing a negatively curved metric. This paper is substantially different from our previous work [8] in several counts. The first is that the explicit description of the Kobayashi metric for E_λ, $0 < \lambda < 1/2$, given by Pflug and Zwonek [20] is significantly different from the convex Thullen domain case ([6], [12]).

For instance, the boundary of the Kobayashi indicatrix at any point of E_λ is not differentiable, when $0 < \lambda < 1/2$. Thus a great care has been given in order to obtain an explicit description and the regularity of the Wu metric. Also, in showing that the holomorphic curvature of the Wu metric is negative in the sense of currents at the points where the Wu metric is only of the Hölder class C^α $(0 < \alpha < 1)$, the methods used in [8] are not applicable here. We use a precise comparison between the Wu metric and the pull-back of the Poincaré-Bergman metric of the unit ball to obtain the curvature property.

The exposition of this paper is self-contained. The properties of the Wu metric on a C^∞ smooth strictly convex domain will be discussed in a separate paper.

2. Main theorems

The first main result is the following explicit formula of the Wu metric.

Theorem 1. *The Wu metric of the Thullen domain E_λ with $\lambda < 1/2$ in the coordinates $(p_1, p_2; v_1, v_2)$ for $E_\lambda \times \mathbb{C}^2 = TE_\lambda$ is given by*

$$\begin{aligned} h_m = &\left(\frac{1}{(1-|p_1|^2)(1-|p_1|^2-|p_2|^{2\lambda})} + \frac{(1-|p_1|^2)^{-2+1/\lambda}|p_1|^2|p_2|^2}{\lambda^2((1-|p_1|^2)^{1/\lambda}-|p_2|^2)^2} \right) dp_1 \otimes d\bar{p}_1 \\ &+ \frac{(1-|p_1|^2)^{-1+1/\lambda}\bar{p}_1 p_2}{\lambda((1-|p_1|^2)^{1/\lambda}-|p_2|^2)^2} dp_1 \otimes d\bar{p}_2 \\ &+ \frac{(1-|p_1|^2)^{-1+1/\lambda} p_1 \bar{p}_2}{\lambda((1-|p_1|^2)^{1/\lambda}-|p_2|^2)^2} dp_2 \otimes d\bar{p}_1 \\ &+ \frac{(1-|p_1|^2)^{1/\lambda}}{((1-|p_1|^2)^{1/\lambda}-|p_2|^2)^2} dp_2 \otimes d\bar{p}_2. \end{aligned}$$

It is remarkable to see from this that the Wu metric has the same form for E_λ for both cases when $\lambda < 1/2$ and $\lambda \geq 1/2$, especially because the Kobayashi metric in these two cases do not take the same form. See [6] and [20]. Furthermore, we can easily see from above that

Corollary 1. *The Wu metric of the Thullen domain* E_λ *is real analytic everywhere except at points in the subset* $Z_0 = \{(p_1, 0) \in E_\lambda\}$ *of complex codimension one.*

In contrast to the non-differentiability of the Kobayashi metric (even away from the zero section of the tangent bundle [20]), Corollary 1 shows that the Wu metric is very smooth almost everywhere while it still enjoys the distance decreasing property.

From a differential geometric point of view, the distance decreasing property is usually due to the negativity of the curvature of the metric. See [1], [19], [21], [22], [25] for backgrounds. Therefore, it is natural to investigate such curvature property of the Wu metric. Indeed, our next main result addresses this problem.

Theorem 2. *For* $0 < \lambda < 1/2$*, the Wu metric is complete and its holomorphic curvature in* E_λ *is bounded above by* $-1/2$ *in the sense of currents.*

Notice that this theorem provides a natural answer to the following conjecture of S. Kobayashi for the non-convex Thullen domains. (The same conclusion for the convex Thullen domains has been given in [8].)

Conjecture ([16], [24])**.** *On every complex manifold whose Kobayashi metric is complete and proper, there exists a* C^k *smooth* (for some $k \geq 0$) *complete* (invariant) *Hermitian metric such that its holomorphic curvature is bounded above by a negative constant in the sense of currents.*

When the draft of this article was completed, we became aware of the lectures by J.P. Demailly presented in the Summer Research Institute of the American Mathematical Society in 1995, which introduced a necessary condition for the existence of negatively curved Hermitian metrics on compact complex manifolds, apart from Kobayashi hyperbolicity. However, we have observed that for the bounded domains of holomorphy which we focus upon, there still are no known obstructions for the existence of negatively curved complete Hermitian metrics.

3. Fundamental properties of the Wu metric

The construction of the Wu metric is discussed in detail in Wu [24] and Cheung-Kim [8]. The readers can refer to those articles for details. Here we will summarize the main properties of the Wu metric. It is an upper semicontinuous invariant pseudo-Hermitian metric defined on *every* complex manifold. In particular, if the complex manifold is hyperbolic in the sense of Kobayashi, meaning that the Kobayashi distance is a proper distance, then the Wu metric has the following important additional properties:

Proposition 1 (Wu [24])**.** *Let* M, N *be Kobayashi hyperbolic complex manifolds with the Wu metrics* h_M *and* h_N*, respectively. Then the following hold:*

(1) *The Wu metric is an invariant metric, in the sense that* $F^*h_N = h_M$ *for any biholomorphic mapping* $F : M \to N$.

(2) *The Wu metric is a continuous positive definite Hermitian metric.*

(3) *The Wu metric is complete whenever the Kobayashi distance is a complete distance.*

(4) *If* $f : M \to N$ *is a holomorphic mapping and* $\dim M = n$*, then* $f^*h_N \leq \sqrt{n}\, h_M$.

The following proposition relates the Wu metric and the Kobayashi metric. We will use this description to find the expression of the Wu metric in Theorem 1. First let us fix $x \in M$. Endow any Hermitian inner product on T_xM, and introduce the volume form accordingly. Let k_M denote the Kobayashi-Royden infinitesimal metric on M. Then, for each positive definite Hermitian inner product α on T_xM, consider the unit ball

$$B_\alpha = \{v \in T_xM \mid \alpha(v,v) \leq 1\}.$$

Likewise, denote by

$$\mathcal{K}_x = \{v \in T_x \mid k_M(x;v) \leq 1\},$$

which is commonly called the *Kobayashi indicatrix of M at x*. Then, we have

Proposition 2. (Wu [24]) *The Wu metric h_x on T_xM satisfies*

(1) $\mathcal{K}_x \subseteq B_{h_x}$, *and*

(2) volume of $B_{h_x} \leq$ volume of B_g *for every $g \in \mathcal{P}_x$ that satisfies $\mathcal{K}_x \subseteq B_g$, where the volume is measured by any Hermitian inner product on T_xM and $\mathcal{P}_x$ is the set of all positive definite Hermitian inner products on T_xM.*

In fact, the Wu metric on a Kobayashi hyperbolic manifold can also be determined by its unit ball in each tangent space, which is the unique complex ellipsoid centered at the origin with smallest possible volume among those containing the Kobayashi indicatrix. In this article, we call this ellipsoid the *best fitting ellipsoid*.

4. Best fitting ellipsoid of the Kobayashi indicatrix

Recall that the automorphism group $\operatorname{Aut} E_\lambda$ of E_λ is a real four-dimensional group generated by the biholomorphic automorphisms given by

$$(z_1, z_2) \mapsto \left(e^{i\theta} \frac{z_1 - p}{1 - \bar{p} z_1}, e^{i\psi} z_2 \left(\frac{\sqrt{1-|p|^2}}{1 - \bar{p} z_1} \right)^{1/\lambda} \right)$$

for $\theta, \psi \in \mathbb{R}$ and $p \in \mathbb{C}$ with $|p| < 1$. It is easy to observe that

$$E_\lambda = \{f(0,b) \mid 0 \leq b < 1, f \in \operatorname{Aut} E_\lambda\}.$$

The study of various invariant metrics in E_λ can now be concentrated on the points $(0,b)$ with $0 \leq b < 1$. Let us recite the result of Pflug-Zwonek [20], which describes the Kobayashi metric of the non-convex Thullen domain. By identifying TE_λ with $E_\lambda \times \mathbb{C}^2$, we denote the square of the Kobayashi metric at the point $(0,b)$ along the vector (v_1, v_2) by $k^2((0,b);(v_1,v_2))$.

Theorem 3 (Pflug-Zwonek [20])**.** *Let $w = \left(\frac{b|v_1|}{\lambda|v_2|}\right)^2$ and suppose that $v_2 \neq 0$.*

(1) *If $w \leq 1$, then the square of the Kobayashi metric $k^2((0,b);(v_1,v_2))$ is given by*

$$k^2((0,b);(v_1,v_2)) = \frac{\lambda^2 |v_2|^2}{b^2} \left(\frac{x^{2\lambda-1}}{(1-\lambda)x^{2\lambda} + m x^{2\lambda-2} - b^{2\lambda}} \right)^2, \tag{1}$$

where

$$t = \frac{2m^2 w}{1 + 2\lambda(\lambda-1)w + \sqrt{1 + 4\lambda(\lambda-1)w}} \tag{2}$$

and x in $[0,1]$ is the unique solution of

$$x^{2\lambda} - tx^{2\lambda-2} - (1-t)b^{2\lambda} = 0. \tag{3}$$

(2) *If $w \geq \frac{1}{4\lambda(1-\lambda)}$, then*

$$k^2((0,b);(v_1,v_2)) = \frac{|v_1|^2}{1-b^{2\lambda}} + \frac{b^{2\lambda-2}\lambda^2|v_2|^2}{(1-b^{2\lambda})^2}. \tag{4}$$

(3) *Denote by $k_1^2((0,b);(v_1,v_2))$ and $k_2^2((0,b);(v_1,v_2))$ the right-hand sides of* (1) *and* (4) *above, respectively. If $1 < w < \frac{1}{4\lambda(1-\lambda)}$, then*

$$k^2((0,b);(v_1,v_2)) = \min(k_1^2((0,b);(v_1,v_2)), k_2^2((0,b);(v_1,v_2))).$$

Also, there exists w_0 such that

$$\text{if } w \leq w_0, \text{then } k^2((0,b);(v_1,v_2)) = k_1^2((0,b);(v_1,v_2));$$
$$\text{if } w \geq w_0, \text{then } k^2((0,b);(v_1,v_2)) = k_2^2((0,b);(v_1,v_2)).$$

When $v_2 = 0$, we have

$$k^2((0,b);(v_1,0)) = \frac{|v_1|^2}{(1-b^{2\lambda})}. \tag{5}$$

Let the unit sphere of the Wu metric in the tangent plane $T_{(0,b)}E_\lambda$ be represented by

$$r_1|v_1|^2 + r_2|v_2|^2 + 2\operatorname{Re} r_3 v_1 \bar{v}_2 = 1,$$

where $r_1, r_2 > 0$ and $r_3 \in \mathbb{C}$. Notice that this ellipsoid is *not* invariant under the action of the group

$$T^2 = \{(v_1,v_2) \mapsto (e^{i\theta}v_1, e^{i\psi}v_2) \,|\, \theta, \psi \in \mathbb{R}\}$$

unless $r_3 = 0$. Since the Kobayashi indicatrix $k^2 \leq 1$ at $(0,b)$ is invariant under the action by T^2 above, and since this action is Euclidean volume preserving, the indicatrix would admit more than one best fitting ellipsoid if $r_3 \neq 0$. This contradicts the uniqueness of the Wu metric which is in this context the uniqueness of the best fitting ellipsoid. (See also [13].) We thus have

Lemma 2. *In terms of the Euclidean coordinates on the tangent bundle $TE_\lambda = E_\lambda \times \mathbb{C}^2$ of the Thullen domain E_λ, for every $(0,b) \in E_\lambda$ the unit sphere of the Wu metric in $T_{(0,b)}E_\lambda$ is described by*

$$r_1|v_1|^2 + r_2|v_2|^2 = 1,$$

where r_1 and r_2 are positive real-valued continuous functions of b.

We will now use the concept of *square convexity* introduced in [8] to find the best fitting ellipsoid. It involves several steps.

Step 1: Reduction to the real case. Since the expressions of the Kobayashi indicatrix and the best fitting ellipsoid (Lemma 2) at $(0,b)$ involve terms of the form $|v_1|^2$ and $|v_2|^2$ only, it is enough to consider the situation when $v_1 \geq 0, v_2 \geq 0$. Thus the problem can now be reduced to finding $r_1 > 0$ and $r_2 > 0$ such that the ellipse $r_1 v_1^2 + r_2 v_2^2 = 1$ in the first quadrant encloses the smallest area with the two axes while containing the set $S = \{(v_1,v_2) | v_1 \geq 0, v_2 \geq 0, k^2(v_1,v_2) \leq 1\}$.

Step 2: The square transformation. Consider the transformation

$$T : (v_1, v_2) \mapsto (u_1, u_2) = (v_1^2, v_2^2),$$

which we call the *square transformation.* Under this transformation, all the regular ellipses in $\mathbb{R}^2$ centered at the origin become straight line segments in the first quadrant. The problem of finding the best fitting ellipsoid can now be reduced to finding a line segment $r_1u_1 + r_2u_2 = 1$ in the first quadrant that encloses the smallest area with the two axes and contains the set $T[S]$.

Step 3: Description of the indicatrix under square transformation. The boundary of the set $T[S]$ consists of parts of the two axes and a curve in the first quadrant. This curve is composed of $\{T(v_1, v_2)|v_1 > 0, v_2 > 0, w < w_0, k_1^2(v_1, v_2) = 1\}$ and $\{T(v_1, v_2)|v_1 > 0, v_2 > 0, w \geq w_0, k_2^2(v_1, v_2) = 1\}$, which we call the *upper curve* and the *lower curve,* respectively. From formula (4), the lower curve is described as

$$\frac{u_1}{(1 - b^{2\lambda})} + \frac{b^{2\lambda-2}\lambda^2 u_2}{(1 - b^{2\lambda})^2} = 1, \tag{6}$$

which is an equation of a line. The description of the upper curve is more complicated. Notice from the expression (2) that

$$w = \frac{t}{(\lambda + (1 - \lambda)t)^2}, \tag{7}$$

which means that

$$\frac{\lambda^2|v_2|^2}{b^2} = \frac{|v_1|^2(\lambda + (1 - \lambda)t)^2}{t}. \tag{8}$$

We thus have

$$\begin{aligned} k_1^2((0, b); (v_1, v_2)) &= \frac{\lambda^2|v_2|^2}{b^2}\left(\frac{x^{2\lambda-1}}{(1 - \lambda)x^{2\lambda} + mx^{2\lambda-2} - b^{2\lambda}}\right)^2 \\ &= \frac{|v_1|^2(x^{2\lambda-1})^2}{(x^{2\lambda-2} - b^{2\lambda})(x^{2\lambda} - b^{2\lambda})}. \end{aligned} \tag{9}$$

Since x is uniquely determined by t, it follows by the equation (3) that

$$t = \frac{x^{2\lambda} - b^{2\lambda}}{x^{2\lambda-2} - b^{2\lambda}}. \tag{10}$$

The upper curve can thus be parameterized by x as follows:

$$u_1(x) = \frac{(x^{2\lambda-2} - b^{2\lambda})(x^{2\lambda} - b^{2\lambda})}{(x^{2\lambda-1})^2}, \tag{11}$$

$$u_2(x) = \frac{b^2}{\lambda^2 x^2}\left((1 - \lambda)x^2 + \lambda - x^2\frac{b^{2\lambda}}{x^{2\lambda}}\right)^2, \tag{12}$$

for $b < x < x_0$, where $x_0 < 1$ is uniquely determined by w_0 as mentioned in Theorem 3.

Step 4: Square convexity. By a direct computation, we have

$$u_1'(x) = \frac{-2b^{2\lambda}E}{x^{2\lambda+1}}, \tag{13}$$

$$u_2'(x) = \frac{2b^2 FE}{\lambda^2 x^3}, \tag{14}$$

where

$$E = -\lambda - (\lambda-1)x^2 + (2\lambda-1)x^2\frac{b^{2\lambda}}{x^{2\lambda}} \tag{15}$$

and

$$F = (1-\lambda)x^2 + \lambda - x^2\frac{b^{2\lambda}}{x^{2\lambda}}. \tag{16}$$

To study the convexity of the upper curve, we first need to show that F is > 0 and E is < 0 for every x with $b < x < x_0$. Since $u_2 > 0$, it is easy to see that F is also > 0 from the expressions (12) and (16). To show that E is < 0 for $b < x < x_0$, we first notice that $E(b) < 0$ and $E' > 0$ for $b < x < 1$. Also, it is clear that u_1' and u_2' have opposite signs and $u_1 > 0$, $u_2 > 0$. From the definition of w and the expressions (7) and (10), we observe that $\frac{u_1(x)}{u_2(x)}$ is a one-to-one function of x for $b < x < x_0$. Now, suppose $E = 0$ at $\tilde{x}$ with $\tilde{x} < x_0$. Then $\frac{u_1(x)}{u_2(x)}$ is an increasing function of x in the range $b < x < \tilde{x}$ and a decreasing function in the range $\tilde{x} < x < x_0$. This obviously contradicts the injectivity of u_1/u_2. We thus have $E < 0$ for $b < x < x_0$.

Now we are ready to study the convexity of the upper curve.

Lemma 3. *Suppose the upper curve is described by $u_2 = f(u_1)$; then f is a convex function.*

Proof. It suffices to show that $(u_1'u_2'' - u_2'u_1'')/(u_1')^3 > 0$. From the previous discussion, we know that $u_1' > 0$. By a direct computation we have

$$\begin{aligned} u_1'u_2'' - u_2'u_1'' &= \frac{2b^{2\lambda}E^2(2b^2)}{x^{2\lambda+1}\lambda^2}\left(\frac{-xF' - 2mF + 2F}{x^4}\right) \\ &= \frac{8b^{2\lambda+2}E^2}{x^{2\lambda+1}\lambda}\frac{(1-x^2)(1-\lambda)}{x^4} > 0. \end{aligned}$$

□

Step 5: Best fitting ellipsoid. Notice that the u_1-intercept of the lower curve is $(1-b^{2\lambda})$ and that the u_2-intercept of the upper curve is $(1-b^2)^2$. Let ℓ be the line segment joining these two intercepts. It is easy to check that the slope of ℓ is larger than the slope of the lower curve. Now, using Lemma 3, we can then conclude that this line segment ℓ together with the two axes in the first quadrant will enclose the smallest area containing the set $T[S]$. This line segment ℓ will correspond to the best fitting ellipsoid under square transformation. We can thus conclude that the

Wu metric at $(0,b)$ is given by:

$$\begin{aligned} h_{1\bar{1}}(0,b) &= \frac{1}{1-b^{2\lambda}}, \\ h_{1\bar{2}}(0,b) &= h_{2\bar{1}}(0,b) = 0, \\ h_{2\bar{2}}(0,b) &= \frac{1}{(1-b^2)^2}. \end{aligned}$$

Now, using the invariance of the Wu metric under the action of the automorphisms of E_λ such as

$$(z_1, z_2) \mapsto \left(\frac{z_1 - p_1}{1 - \bar{p}_1 z_1}, \frac{|p_2|}{p_2} \cdot \frac{(1-|p_1|^2)^{1/2\lambda}\ z_2}{(1-\bar{p}_1 z_1)^{1/\lambda}} \right) \tag{17}$$

which maps (p_1, p_2) with $p_2 \neq 0$ to $(0, |p_2|(1-|p_1|^2)^{-1/2\lambda})$, we arrive at the explicit formula of the Wu metric tensor stated in Theorem 1. (The case when $p_2 = 0$ will also be included in the explicit formula by the continuity of the Wu metric.)

Notice that the expression of the Wu metric in Theorem 1 is exactly the same as in the case of convex Thullen domain E_λ with $1/2 \leq \lambda < 1$ ([8]). When $\lambda = 1$, E_1 is the unit ball, and the Wu metric agrees with the Poincaré-Bergman metric. Also one can see from the expression that the Wu metric is real analytic everywhere except when $p_2 = 0$.

5. Negative holomorphic curvature

5.1. Notation and generalities. Let g be a C^2 Hermitian metric on a complex manifold M with $g = \sum g_{i\bar{j}} dz^i d\bar{z}^j$ in local coordinates; then the coefficients of the curvature tensor are given by

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} + \sum g^{\alpha\bar{\beta}} \frac{\partial g_{i\bar{\beta}}}{\partial z^k} \frac{\partial g_{\alpha\bar{j}}}{\partial \bar{z}^\ell}.$$

The holomorphic curvature at a point p in the direction $(\zeta^1, \zeta^2, \ldots, \zeta^n)$ is

$$\frac{\sum R_{i\bar{j}k\bar{\ell}}(p)\zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell}{\sum g_{i\bar{j}}(p) g_{k\bar{\ell}}(p) \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell}.$$

The *holomorphic curvature is said to be bounded from above by a negative constant* $-c^2$, if

$$\sum R_{i\bar{j}k\bar{\ell}}(p)\zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell < -c^2 \sum g_{i\bar{j}}(p) g_{k\bar{\ell}}(p) \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell$$

for every direction $(\zeta^1, \zeta^2, \ldots, \zeta^n)$. (See [17] for instance.) If g is only continuous, the above definition can still have a well-defined meaning as explained in [24]: the holomorphic curvature of g is *bounded from above by a negative constant* $-c^2$ *in the sense of currents*, if for every embedded Riemann surface $S \subset M$, let $g|_S = G dz d\bar{z}$, then $\partial\bar{\partial} \log G \geq c^2 G dz \wedge d\bar{z}$ in the sense of currents. If g is C^2, the two notions of negativity introduced here coincide.

Now, we are ready to check if the Wu metric possesses negative holomorphic curvature.

5.2. Curvature at the smooth point. Since the Wu metric is an invariant metric, it suffices to check the curvature at the point $(0, b)$. In this section, we look at the case when $b > 0$, that is where the Wu metric is real-analytic. The computation here is formally the same as that of the case $1/2 \leq \lambda < 1$ that is presented in [8]. Nevertheless, we will outline the computation briefly here for the sake of completeness. By a direct computation, we have

$$\sum R_{i\bar{j}k\bar{\ell}}(0,b)\zeta^i\bar{\zeta}^j\zeta^k\bar{\zeta}^\ell = \left(\frac{-1}{(1-b^{2\lambda})^2} + \frac{-b^2}{(1-b^2)^2\lambda^2} - \frac{1}{1-b^{2\lambda}}\right)|\zeta^1|^4$$
$$+ \left(\frac{-b^{2\lambda}\lambda^2}{(1-b^{2\lambda})^3 b^2} + \frac{-3b^2-3}{(1-b^2)^3\lambda} + \frac{2b^{2\lambda}}{(1-b^{2\lambda})(1-b^2)^2} + \frac{(1-b^{2\lambda})b^2}{(1-b^2)^4\lambda^2}\right)|\zeta^1|^2|\zeta^2|^2$$
$$+ \frac{-2}{(1-b^2)^4}|\zeta^2|^4.$$

Using the inequalities

$$\lambda b^\lambda(1-b^2) < 1 - b^{2\lambda} < \lambda(1-b^2)b^{\lambda-1} \tag{18}$$

for $0 < b < 1$ and $0 < \lambda < 1$, we have

$$\frac{-b^{2\lambda}\lambda^2}{(1-b^{2\lambda})^3 b^2} + \frac{2b^{2\lambda}}{(1-b^{2\lambda})(1-b^2)^2} + \frac{-3b^2-3}{(1-b^2)^3\lambda} + \frac{(1-b^{2\lambda})b^2}{(1-b^2)^4\lambda^2}$$
$$< \frac{-\lambda^2 b^{2\lambda}}{(1-b^{2\lambda})^3 b^2} + \frac{2}{(1-b^{2\lambda})(1-b^2)^3\lambda}[-(1-b^{2\lambda}) + \lambda b^{2\lambda}(1-b^2)] < 0.$$

This means that

$$\sum R_{i\bar{j}k\bar{\ell}}(0,b)\zeta^i\bar{\zeta}^j\zeta^k\bar{\zeta}^\ell$$
$$\leq \left(\frac{-1}{(1-b^{2\lambda})^2} + \frac{-b^2}{(1-b^2)^2\lambda^2} - \frac{1}{1-b^{2\lambda}}\right)|\zeta^1|^4 + \frac{-2}{(1-b^2)^4}|\zeta^2|^4$$
$$\leq -\frac{1}{2}\sum h_{i\bar{j}}(0,b)h_{k\bar{\ell}}(0,b)\zeta^i\bar{\zeta}^j\zeta^k\bar{\zeta}^\ell.$$

We can thus conclude that

Lemma 4. *The holomorphic curvature of the Wu metric in E_λ is bounded above by $-1/2$ at every point where the Wu metric is smooth.*

5.3. Negative curvature in the sense of current. We now check the curvature of the Wu metric at $(0, 0)$. We first need comparison of the Wu metric with the Bergman metric of the unit ball.

Lemma 5. *Let β be the Poincaré-Bergman metric of the unit ball E_1 in $\mathbb{C}^2$ and i the inclusion mapping from E_λ to E_1, with $0 < \lambda < 1/2$. Denote the Wu metric on E_λ by h. Then there exists a small neighborhood of $(0, 0)$ such that $i^*\beta \leq h$ at every point in the neighborhood and $i^*\beta(0, 0) = h(0, 0)$.*

Proof. Property (4) of Proposition 1 implies immediately that $i^*\beta \leq \sqrt{2}h$. But we need a sharper inequality. Denote the component of the metric $i^*\beta$ by $g_{i\bar{j}}$. Then by a direct computation, we have at any point $(p_1, p_2) \in E_\lambda$,

$$h_{1\bar{1}} - g_{1\bar{1}} = \frac{(|p_2|^{2\lambda} - |p_2|^2)}{(1-|p_1|^2)(1-|p_1|^2-|p_2|^{2\lambda})(1-|p_1|^2-|p_2|^2)}, \tag{19}$$

$$h_{2\bar{2}} - g_{2\bar{2}} = \frac{((1-|p_1|^2)^{1+1/\lambda} - |p_2|^4)(1-(1-|p_1|^2)^{-1+1/\lambda})(1-|p_1|^2)}{(1-|p_1|^2-|p_2|^2)\left((1-|p_1|^2)^{1/\lambda} - |p_2|^2\right)^2}, \tag{20}$$

$$\begin{aligned}(h_{1\bar{1}} - g_{1\bar{1}})&(h_{2\bar{2}} - g_{2\bar{2}}) - (h_{1\bar{2}} - g_{1\bar{2}})(h_{2\bar{1}} - g_{2\bar{1}}) \\ &= \frac{E_1 - E_2}{\left((1-|p_1|^2)^{1/\lambda} - |p_2|^2\right)^2 (1-|p_1|^2-|p_2|^2)^2}\end{aligned} \tag{21}$$

where

$$E_1 = \frac{(|p_2|^{2\lambda} - |p_2|^2)((1-|p_1|^2)^{1+1/\lambda} - |p_2|^4)(1-(1-|p_1|^2)^{-1+1/\lambda})}{(1-|p_1|^2-|p_2|^{2\lambda})(1-|p_1|^2-|p_2|^2)}$$

and

$$E_2 = |p_1|^2|p_2|^2(1-|p_1|^2)^{-1+1/\lambda}\left(\frac{1}{\lambda} - 1\right)^2.$$

Notice that the expressions (19), (20) and (21) are ≥ 0 when $|p_1|$ and $|p_2|$ are small enough. The assertion of the lemma therefore follows immediately. □

Let $Z_0 = \{(p_1, 0) \in E_\lambda\}$ be the set of points in E_λ where the Wu metric is not differentiable. Let $S \in E_\lambda$ be an embedded Riemann surface with complex coordinate $\{s\}$ that passes through $(0,0)$ at $s = 0$. Suppose that $h|_S = H ds d\bar{s}$, and $i^*\beta|_S = G ds d\bar{s}$; we want to show that $\partial\bar{\partial} \log H \geq (1/2) H ds \wedge d\bar{s}$ in the sense of currents. In view of Lemma 4, it suffices to show that for every $z \in S \cap Z_0$,

$$\lim_{r \to 0} \frac{\int_{B(z,r)} (\log H(s) - \log H(z)) dV(s)}{\int_{B(z,r)} (s-z)(\bar{s}-\bar{z}) dV(s)} \geq \frac{H(z)}{2}, \tag{22}$$

where dV denotes the Euclidean volume form. By Lemma 5, there exists a small neighborhood $B(0,r) \cap S$, such that

$$H(s) \geq G(s), \qquad H(0) = G(0). \tag{23}$$

Also by the real analyticity of the metric $i^*\beta$ we have

$$\lim_{r \to 0} \frac{\int_{B(0,r)} (\log G(s) - \log G(0)) dV}{\int_{B(0,r)} s\bar{s} dV} = \frac{\partial^2 \log(G(s))}{\partial s \partial \bar{s}}. \tag{24}$$

Since the Poincaré-Bergman metric of the ball has constant holomorphic curvature -2, by the decreasing holomorphic curvature property of submanifolds ([17] and [23]) we have

$$\frac{\partial^2 \log(G(s))}{\partial s \partial \bar{s}} \geq 2G(0). \tag{25}$$

Expressions (23), (24) and (25) immediately imply the inequality (22) in the case $z = 0$. At the non-smooth point $(p_1, 0) \in S \cap Z_0$ with $p_1 \neq 0$, one can still prove inequality (22) following the previous arguments, except for one technical point that we need to replace the metric $i^*\beta$ by $\mu^* i^*\beta$ in Lemma 5, where μ is the Möbius transformation

$$(z_1, z_2) \mapsto \left(\frac{z_1 - p_1 a}{1 - \bar{p}_1 z_1}, \frac{(1 - |p|^2)^{1/2\lambda}}{(1 - \bar{p}_1 z_1)^{1/\lambda}} z_2 \right),$$

which is an automorphism of E_λ and maps $(p_1, 0)$ to $(0, 0)$. This then completes the proof of Theorem 2.

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