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# ON THE NON-EXISTENCE OF HOMOCLINIC ORBITS FOR A CLASS OF INFINITE DIMENSIONAL HAMILTONIAN SYSTEMS

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ABSTRACT. We prove that for a class of infinite dimensional Hamiltonian systems certain homoclinic connections to the origin cease to exist when the nonlinearities have 'super-critical' growth. The proof is based on a variational principle and a Pohožaev type identity.

## 1. INTRODUCTION

In a recent paper Clément, Felmer and Mitidieri [2] studied the existence of homoclinic orbits of the following infinite dimensional Hamiltonian system:

$$\text{(H)}\begin{cases} -\frac{\partial v}{\partial t} - \Delta v = |u|^{p-1}u, & \text{in } \Omega\times\mathbf{R}, \quad (1.1)\\ \frac{\partial u}{\partial t} - \Delta u = |v|^{q-1}v, & \text{in } \Omega\times\mathbf{R}, \quad (1.2)\\ u(x,t)=v(x,t)=0, & \text{on } \partial\Omega\times\mathbf{R}, \quad (1.3)\\ \lim_{t\to\pm\infty} u(t,x) = \lim_{t\to\pm\infty} v(t,x) = 0, & \text{uniformly in } x\in\Omega, \quad (1.4)\end{cases}$$

where $\Omega\subset\mathbf{R}^n$, $n\geq 1$, is a bounded domain with smooth boundary $\partial\Omega$ and $pq>1$, $p,q>0$. Homoclinic connections to the origin are solutions $u,v\in C^{2,1}(\overline{\Omega}\times\mathbf{R})$. These solutions will also be referred to as classical solutions of System (H). The main result in [2] is:

**Theorem 1.1.** *Assume that $p,q$ satisfy*

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{n}{n+2}, \quad p,q>0. \tag{1.5}$$

*Then System (H) has at least one non-trivial classical solution (homoclinic connection to the origin) with positive components.*

In this note we shall prove that (1.5) is indeed 'optimal' for the existence of homoclinic connections to the origin. When (1.5) does not hold and $\Omega$ is star-shaped, System (H) does not have classical solutions with positive components. Our main result can be stated as follows:

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**Theorem 1.2.** *Let $\Omega \subset \mathbf{R}^n$ be star-shaped and assume that $p, q$ satisfy*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n}{n+2}, \quad p, q > 0. \tag{1.6}$$

*Then System (H) has no homoclinic connections to the origin with positive components.*

The relations (1.5) and (1.6) are complementary, which proves the sharpness of both Theorem 1.1 and 1.2 in the case when $\Omega$ is star-shaped. The case of nonstar-shaped domains is much more delicate and will not be considered here (e.g., see [1] for elliptic equations).

Observe that the hyperbola of critical exponents

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n}{n+2}, \quad p, q > 0, \tag{1.7}$$

is different from the one for the stationary equations (e.g., see [5], [12]), since our domain is $\Omega \times \mathbf{R}$ instead of $\Omega$. If one considers the case $q = 1$ one can rewrite the equations (1.1) and (1.2) into a hypo-elliptic non-homogeneous 2nd/4th order equation in the strip $\Omega \times \mathbf{R}$. For this equation the theory of Pucci-Serrin [9] can be applied and gives the same critical exponent as relation (1.7) for $q = 1$.

The equations (1.1) and (1.2) can be derived from a variational principle, i.e. they can be viewed as the Euler-Lagrange equations of the functional

$$J(u, v) = \int_{\mathbf{R}} \int_{\Omega} \Big( v \frac{\partial u}{\partial t} + \nabla u \nabla v - \frac{1}{p+1}|u|^{p+1} - \frac{1}{q+1}|v|^{q+1} \Big) dx dt, \tag{1.8}$$

where the integrand will be denoted by $\mathcal{J}$. The integrand $\mathcal{J}$ is called the Lagrangian density of System (H). Observe that if $\Omega = \mathbf{R}^n$ and the $p, q$ satisfy (1.7), then the functional $J$ is invariant under the transformations

$$u(x,t) \longrightarrow e^{-\frac{n+2}{p+1}\varepsilon} u\left(e^{-\varepsilon}x, e^{-2\varepsilon}t\right), \quad v(x,t) \longrightarrow e^{-\frac{n+2}{q+1}\varepsilon} v\left(e^{-\varepsilon}x, e^{-2\varepsilon}t\right), \tag{1.9}$$

with $\varepsilon \in (\mathbf{R}, +)$. In this situation the celebrated Noether's Theorem [6] provides a conservation law for System (H). However our system is not invariant under the transformations given by (1.9). The adaptation of Noether's Theorem as described in [4] and [12] (see also [5] and [9]) will be used to obtain an almost conservation law, which can be interpreted as a Pohožaev type identity (see [8], [12]).

Since System (H) is autonomous, the functional $J$ is invariant under time-translations, i.e. the transformations

$$u(x,t) \longrightarrow u(x, t+\tau), \quad v(x,t) \longrightarrow v(x, t+\tau), \quad \tau \in \mathbf{R}, \tag{1.10}$$

leave $J$ invariant. Again from Noether's Theorem one obtains a conservation law, given by

$$H(u,v) = \int_{\Omega} \Big( \nabla u \nabla v - \frac{1}{p+1}|u|^{p+1} - \frac{1}{q+1}|v|^{q+1} \Big) dx = \text{constant}, \tag{1.11}$$

which shall be referred to as the *Hamiltonian* of System (H).

If one replaces the boundary conditions (1.3) by $u(x,t) = \frac{\partial u}{\partial n}(x,t) = 0$ on $\partial\Omega \times \mathbf{R}$, one can prove that System (H) does not have homoclinic connections when $\frac{1}{p+1} + \frac{1}{q+1} < \frac{n}{n+2}$ and $\Omega$ is star-shaped. In order to handle the critical case one needs a unique continuation result for the linear differential operator in System (H).

## 2. Almost invariance and Noether's Theorem

As was mentioned in Section 1, the transformations (1.9) only leave $J$ invariant when $p, q$ satisfy (1.7). Invariance means that the 'Lie-derivative' of $J$ with respect to this transformation group is zero at (classical) solutions of System (H). In the case that (1.7) is not satisfied one can still compute the Lie-derivative of $J$ with respect to the transformation group given by (1.9).

In order to do so we shall now use the Lagrangian density $\mathcal{J}(u, v, \nabla u, \nabla v, u_t, v_t)$. The variables $u, v$ $x^i$ and $t$ will be considered as the independent variables, and $p_u^i = \nabla u$, $p_v^i = \nabla v$, $p_u^t = u_t$ and $p_v^t = v_t$ as the dependent variables. The transformations given by (1.9) are equivalent to the following set of transformations of the independent variables $u, v, x^i, t$:

$$\tilde{u} = e^{-\alpha\varepsilon} u, \quad \tilde{x}^i = e^{\varepsilon} x^i, \tag{2.1}$$

$$\tilde{v} = e^{-\beta\varepsilon} v, \quad \tilde{t} = e^{2\varepsilon} t, \tag{2.2}$$

where $\alpha = \frac{n+2}{p^*+1}$, $\beta = \frac{n+2}{q^*+1}$ and $p^*, q^*$ satisfy (1.7). We write $g_\varepsilon(u, v, x^i, t) = (\tilde{u}, \tilde{v}, \tilde{x}^i, \tilde{t})$. The infinitesimal generator of $g_\varepsilon$ is given by (see e.g. [4], [7])

$$\mathbf{v} = x^i \frac{\partial}{\partial x^i} + 2t\frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u} - \beta v \frac{\partial}{\partial v}. \tag{2.3}$$

The way the variables $p_u^i, p_v^i, p_u^t$ and $p_v^t$ transform now will depend on (2.1) and (2.2). The Lie-derivative of $\mathcal{J}$ with respect to $g_\varepsilon$ is defined by

$$D_{g_\varepsilon}\mathcal{J} = \frac{d}{d\varepsilon}\Big[\mathcal{J}(g_\varepsilon(u, v, x^i, t))\frac{d\tilde{x}}{dx}\frac{d\tilde{t}}{dt}\Big]\Big|_{\varepsilon=0}, \tag{2.4}$$

where $\frac{d\tilde{x}}{dx} = \det\big(\frac{\partial \tilde{x}^i}{\partial x^i}\big)$. If $p_u^i, p_v^i, p_u^t$ and $p_v^t$ were independent variables (which would not be transformed), one can compute the Lie-derivative of $\mathcal{J}$ at a classical solution by simply applying the differential operator $\mathbf{v}$, given by (2.3), to $\mathcal{J}$. However $\mathcal{J}$ depends also on the dependent variables $p_u^i, p_v^i, p_u^t$ and $p_v^t$, so that one needs to extend the infinitesimal generator $\mathbf{v}$ in order to cope with the dependent variables. The extension of $\mathbf{v}$ is denoted by $\mathbf{v}^{(1)}$ and is called the prolonged vectorfield (see [7] for an explicit formula). The expression for the Lie-derivative now becomes

$$D_{g_\varepsilon}\mathcal{J} = \mathbf{v}^{(1)}\mathcal{J} + (n+2)\mathcal{J}. \tag{2.5}$$

Applying the transformations (2.1) and (2.2) to $\mathcal{J}$ directly and computing $D_{g_\varepsilon}\mathcal{J}$ according to (2.4) obviously gives the same result. One may choose either of these two ways to compute $D_{g_\varepsilon}\mathcal{J}$.

**Lemma 2.1.** *Let $u, v \in C^{2,1}(\overline{\Omega} \times \mathbf{R})$ be a classical solution of System (H). Then*

$$D_{g_\varepsilon}\mathcal{J} = \big(\alpha - \frac{n+2}{p+1}\big)|u|^{p+1} + \big(\beta - \frac{n+2}{q+1}\big)|v|^{q+1}, \ \forall (x,t) \in \Omega \times \mathbf{R}, \tag{2.6}$$

*with $\alpha + \beta = n$.*

*Proof of Lemma 2.1.* In order to prove this lemma we can apply Theorem 2.1 of [12] using the infinitesimal generator $\mathbf{v}$. Instead we will compute $\mathcal{J}(g_\varepsilon(u, v, x^i, t))\frac{d\tilde{x}}{dx}\frac{d\tilde{t}}{dt}$

and determine its derivative at $\varepsilon = 0$. Using (2.1) and (2.2), we find that

$$\mathcal{J}(g_\varepsilon(u,v,x^i,t))\frac{d\tilde{x}}{dx}\frac{d\tilde{t}}{dt} = v\frac{\partial u}{\partial t} + \nabla u \nabla v - \frac{e^{(-(p+1)\alpha+n+2)\varepsilon}}{p+1}|u|^{p+1} - \frac{e^{(-(q+1)\beta+n+2)\varepsilon}}{q+1}|v|^{q+1},$$

where $\frac{d\tilde{x}}{dx} = e^{n\varepsilon}$. Applying (2.4), we immediately obtain (2.6). □

From Lemma 2.1 it immediately follows that the Lie-derivative $D_{g_\varepsilon}\mathcal{J}$ is positive (positive pointwise in $\Omega \times \mathbf{R}$) when $p, q$ satisfy (1.6) with strict inequality, and identically equal to zero when $p, q$ satisfy (1.7). Indeed if $p, q$ satisfy (1.6) with strict inequality, one can choose numbers $p^*$ and $q^*$, which satisfy (1.7), such that $\frac{1}{p+1} < \frac{1}{p^*+1}$ and $\frac{1}{q+1} < \frac{1}{q^*+1}$. Taking $\alpha = \frac{n+2}{p^*+1}$ and $\beta = \frac{n+2}{q^*+1}$ then gives the desired result. In the case that $p$ and $q$ satisfy (1.7) one chooses $\alpha = \frac{n+2}{p+1}$ and $\beta = \frac{n+2}{q+1}$.

If we now apply the modified version of Noether's Theorem (see [4], [12]), we obtain the local form of a Pohožaev type identity for solutions of System (H).

**Lemma 2.2.** *Let $u, v \in C^{2,1}(\overline{\Omega} \times \mathbf{R})$ be a classical solution of System (H) and let*

$$\mathbf{P} = (P^i, P^t) = (x^i\mathcal{J} + Q_u\frac{\partial \mathcal{J}}{\partial p_u^i} + Q_v\frac{\partial \mathcal{J}}{\partial p_v^i}, 2t\mathcal{J} + Q_u\frac{\partial \mathcal{J}}{\partial p_u^t} + Q_v\frac{\partial \mathcal{J}}{\partial p_v^t}), \tag{2.7}$$

*with $Q_u = -\alpha u - p_u^i x^i - 2tp_u^t$, $Q_v = -\beta v - p_v^i x^i - 2tp_v^t$. Then,*

$$\mathit{div}\,\mathbf{P} = D_{g_\varepsilon}\mathcal{J} = \big(\alpha - \frac{n+2}{p+1}\big)|u|^{p+1} + \big(\beta - \frac{n+2}{q+1}\big)|v|^{q+1},\ \forall (x,t) \in \Omega \times \mathbf{R}. \tag{2.8}$$

*Proof of Lemma 2.2.* See [4] and [12]. □

If one determines the expressions for $P^i$ and $P^t$, it follows that

$$\begin{aligned} P^i &= x^i v\frac{\partial u}{\partial t} + x^i\nabla u\nabla v - \frac{x^i}{p+1}|u|^{p+1} - \frac{x^i}{q+1}|v|^{q+1} \\ &\quad - (x\cdot\nabla u + 2t\frac{\partial u}{\partial t})\nabla v - (x\cdot\nabla v + 2t\frac{\partial v}{\partial t})\nabla u - \alpha u\nabla v - \beta v\nabla u, \\ P^t &= 2tv\frac{\partial u}{\partial t} + 2t\nabla u\nabla v - \frac{2t}{p+1}|u|^{p+1} - \frac{2t}{q+1}|v|^{q+1} \\ &\quad - (x\cdot\nabla u + 2t\frac{\partial u}{\partial t})v - \alpha uv. \end{aligned}$$

In the next section we shall use these expressions in order to obtain an integral identity for solutions of System (H).

## 3. The integral identity and the proof of Theorem 1.2

In order to integrate (2.8) one needs certain a priori integrability properties for classical solutions of System (H). Since $\Omega$ is bounded and the solutions $u, v$ are continuous up to the boundary in $x$, $L^r$-integrability in $x$ for any $r \geq 1$ is easily obtained. However, since the domain of definition of the time-variable is $\mathbf{R}$, the integrability properties are not immediately clear. A precise analysis of the asymptotic behavior of a solution $(u, v)$ near the trivial solution $(0, 0)$ will give us the required information. It follows, if $pq > 1$ (or equivalently $\frac{1}{p+1} + \frac{1}{q+1} < 1$), that $u$ and $v$ converge to zero as $t \to \pm\infty$, uniformly in $x$ with exponential decay in the $t$-variable. For the case $p, q \geq 1$ this is an obvious byproduct of the fact

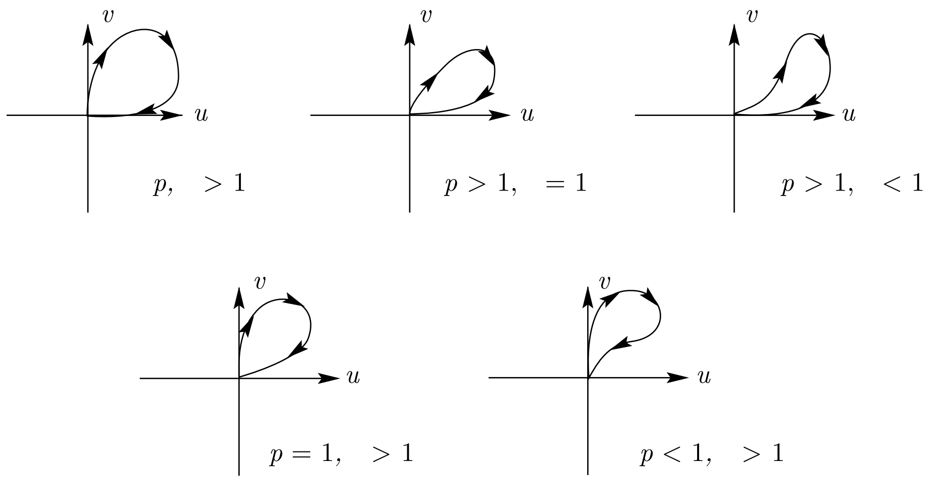


FIGURE 1

that $(u,v)(x,t)$ approach the origin along the local stable and unstable manifold of the origin as $t \to \pm\infty$ respectively (see [3]). However if $p < 1$ ($q > 1$) or if $q < 1$ ($p > 1$), one cannot apply the above result, although the asymptotic behavior is still exponential in time (see Fig. 1).

**Lemma 3.1.** *Let $u, v \in C^{2,1}(\overline{\Omega} \times \mathbf{R})$ be a classical solution of System (H). Then there exist $t_0 > 0$ and constants $C_1, \ldots, C_4 > 0$ such that*

$$\sup_{x\in\overline{\Omega}} |u(x,t)| \le C_1 e^{-\gamma(t-t_0)}, \quad \sup_{x\in\overline{\Omega}} |v(x,t)| \le C_2 e^{-\gamma p(t-t_0)}, \; t \ge t_0, \tag{3.1}$$

*and*

$$\sup_{x\in\overline{\Omega}} |u(x,t)| \le C_3 e^{\gamma q(t+t_0)}, \quad \sup_{x\in\overline{\Omega}} |v(x,t)| \le C_4 e^{\gamma(t+t_0)}, \; -t \ge t_0, \tag{3.2}$$

*for any $0 < \gamma < \lambda_1$, where $\lambda_1$ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions.*

*Proof of Lemma 3.1.* For any $1 < r < \infty$ the operator $-\Delta$ (with Dirichlet boundary conditions) is sectorial on $L^r(\Omega)$ [3] with $D(\Delta) = W^{2,r} \cap W_0^{1,r}(\Omega) = X_1$, and its spectrum $\sigma(-\Delta)$ is contained in the half-plane $\{\lambda;\ Re(\lambda) > \gamma', 0 < \gamma' < \lambda_1\}$, where $\lambda_1$ is the principal eigenvalue of $-\Delta$. The operator $e^{t\Delta}$ satisfies the estimate

$$\|e^{t\Delta} u\|_{L^r} \le M_r e^{-\gamma' t} \|u\|_{L^r},$$

with $0 < \gamma' < \lambda_1$ and $M_r \ge 1$. From the boundedness of $u(t)$ and $v(t)$ in $L^r(\Omega)$ and from the previous estimate on $e^{t\Delta}$ it follows that the solutions of System (H) verify the integral equations

$$u(t) = e^{(t-t_0)\Delta} u(t_0) + \int_{t_0}^{t} e^{(t-s)\Delta} v^q(s) ds, \tag{3.3}$$

$$v(t) = \int_t^\infty e^{(s-t)\Delta} u^p(s) ds, \tag{3.4}$$

for all $t \geq t_0$, where $t_0$ is arbitrary real number. We claim that there exist numbers $\gamma \in (0, \gamma')$, $\rho > 0$ and $t_0 \in \mathbf{R}$ such that the following estimate holds:

$$\|u(t)\|_{L^r} \leq 2M_r e^{-\gamma(t-t_0)}\rho + 2^{-n+1}M_r\rho, \quad \forall t \geq t_0, \ \forall n \geq 1. \tag{3.5}$$

We shall prove (3.5) by induction. First we choose $\gamma \in (0, \gamma')$, and since $pq > 1$, there exists a $\rho > 0$ satisfying

$$\left[5M\left(\frac{M'}{\gamma'}\right)^q \frac{1}{\gamma' - \gamma}\rho^{pq-1}\right] < 1/2, \tag{3.5}$$

where $M = M_r$ and $M' = M_{rq}$. Finally we choose $t_0$ large enough so that $\|u(t)\|_{L^r} \leq \rho$, for $t \geq t_0$. With this of choice $t_0$, $\rho$ and $\gamma$, (3.5) holds for $n = 1$. From (3.3) and (3.4) it follows that

$$\begin{aligned} \|u(t)\|_{L^r} &\leq Me^{-\gamma'(t-t_0)}\|u(t_0)\|_{L^r} + M\int_{t_0}^t e^{-\gamma'(t-s)}\|v^q(s)\|_{L^r}ds \\ &\leq Me^{-\gamma(t-t_0)}\|u(t_0)\|_{L^r} + M\int_{t_0}^t e^{-\gamma'(t-s)}\|v(s)\|_{L^{rq}}^q ds, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \|v(s)\|_{L^{rq}} &\leq M'\int_s^\infty e^{-\gamma'(\tau-s)}\|u^p(\tau)\|_{L^{rq}}d\tau \\ &= M'\int_s^\infty e^{-\gamma'(\tau-s)}\|u^{pq}(\tau)\|_{L^r}^{1/q}d\tau. \end{aligned} \tag{3.7}$$

Notice, since $pq > 1$ and $\|u(t)\|_{L^r} \leq \rho$, that $\|u^{pq}(t)\|_{L^r} \leq \rho^{pq-1}\|u(t)\|_{L^r}$ and therefore (3.7) becomes

$$\|v(s)\|_{L^{rq}}^q \leq \left(\frac{M'}{\gamma'}\right)^q \rho^{pq-1} \sup_{[s,\infty)} \|u(\tau)\|_{L^r}. \tag{3.8}$$

At this point we assume that (3.5) is true for $n$. Combining this with (3.6) and (3.8), we get

(3.9)

$$\begin{aligned} \|u(t)\|_{L^r} &\leq Me^{-\gamma'(t-t_0)}\rho + M\left(\frac{M'}{\gamma'}\right)^q\rho^{pq-1}\int_{t_0}^t e^{-\gamma'(t-s)}\sup_{[s,\infty)}\|u(\tau)\|_{L^r}ds \\ &\leq Me^{-\gamma'(t-t_0)}\rho + 2M^2\left(\frac{M'}{\gamma'}\right)^q\rho^{pq}\int_{t_0}^t e^{-\gamma'(t-s)}e^{-\gamma(s-t_0)}ds \\ &\quad + M^2 2^{-n+1}\left(\frac{M'}{\gamma'}\right)^q\rho^{pq}\int_{t_0}^t e^{-\gamma'(t-s)}ds \\ &= Me^{-\gamma'(t-t_0)}\rho \\ &\quad + \left[2M\left(\frac{M'}{\gamma'}\right)^q\frac{1}{\gamma'-\gamma}\rho^{pq-1}\right]e^{-\gamma(t-t_0)}\rho M \\ &\quad - \left[2M\left(\frac{M'}{\gamma'}\right)^q\frac{1}{\gamma'-\gamma}\rho^{pq-1}\right]e^{-\gamma'(t-t_0)}\rho M \\ &\quad - \left[M\left(\frac{M'}{\gamma'}\right)^q\frac{1}{\gamma'}\rho^{pq-1}\right]e^{-\gamma'(t-t_0)}2^{-n+1}\rho M + \left[M\left(\frac{M'}{\gamma'}\right)^q\frac{1}{\gamma'}\rho^{pq-1}\right]2^{-n+1}M\rho \\ &\leq 2Me^{-\gamma(t-t_0)}\rho + 2^{-n}M\rho, \end{aligned}$$

since $\left[5M\left(\frac{M'}{\gamma'}\right)^q \frac{1}{\gamma'-\gamma}\rho^{pq-1}\right] < 1/2$, which proves (3.5). Letting $n \to \infty$ in (3.5), we obtain the estimate

$$\|u(t)\|_{L^r} \leq 2Me^{-\gamma(t-t_0)}\rho. \tag{3.10}$$

Combining (3.10) with (3.8) gives

$$\|v(s)\|^q_{L^{rq}} \leq 2M\left(\frac{M'}{\gamma'}\right)^q \rho^{pq} e^{-\gamma(s-t_0)}. \tag{3.11}$$

Following [3], we define the family of spaces $X_\alpha = D((-\Delta)^\alpha)$, $\alpha \geq 0$, with $X_\alpha \subset L^r(\Omega)$. For $e^{t\Delta}$ the following estimate holds:

$$\|e^{t\Delta}u\|_{X_\alpha} \leq M'' t^{-\alpha} e^{-\gamma' t}\|u\|_{L^r},$$

which yields

$$\|u(t)\|_{X_\alpha} \leq Me^{-\gamma(t-t_0)}\|u(t_0)\|_{L^r} + M''\int_{t_0}^t (t-s)^{-\alpha} e^{-\gamma'(t-s)}\|v(s)\|^q_{L^{rq}} ds. \tag{3.12}$$

Combining (3.12) and (3.11) gives

$$\|u(t)\|_{X_\alpha} \leq Ce^{-\gamma(t-t_0)}\rho, \quad C > 1.$$

A similar estimate can be obtained for $v$:

$$\|v(t)\|_{X_\alpha} \leq C' e^{-\gamma p(t-t_0)}\rho^p, \quad C' > 1.$$

Finally, by choosing $r > n/2$, $1 > \alpha > n/2r$ we have $X_\alpha \hookrightarrow C^0$, which implies (3.1). The proof of (3.2) follows by interchanging $t$ and $-t$, $u$ and $v$, and $p$ and $q$. □

From the proof of Lemma 3.1 it immediately follows that also

$$\sup_{x\in\overline{\Omega}} |\nabla u(x,t)| \leq C_1' e^{-\gamma(t-t_0)}, \quad \sup_{x\in\overline{\Omega}} |\nabla v(x,t)| \leq C_2' e^{-\gamma p(t-t_0)},\ t \geq t_0, \tag{3.1$'$}$$

and

$$\sup_{x\in\overline{\Omega}} |\nabla u(x,t)| \leq C_3' e^{\gamma q(t+t_0)}, \quad \sup_{x\in\overline{\Omega}} |\nabla v(x,t)| \leq C_4' e^{\gamma(t+t_0)},\ -t \geq t_0, \tag{3.2$'$}$$

for some $t_0 > 0$ and constants $C_1', \ldots, C_4'$, for any $0 < \gamma < \lambda_1$. Indeed, if we choose $r > n$ and $1 > \alpha > 1/2 + n/2r$, it follows that $X_\alpha \hookrightarrow C^1$, which proves the above estimates.

As a direct consequence of Lemma 3.1 and the above gradient estimates we can now integrate (2.8) over the domain $\Omega \times \mathbf{R}$. This will give a global form of the modified Noether's Theorem.

**Lemma 3.2.** *Let $u, v \in C^{2,1}(\overline{\Omega} \times \mathbf{R})$ be a classical solution of System (H). Then*

$$\begin{aligned} &-\int_{\mathbf{R}} \oint_{\partial\Omega} \frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x, n_\Omega) dS dt \\ &\qquad = \int_{\mathbf{R}}\int_\Omega \Big(\big(\alpha - \frac{n+2}{p+1}\big)|u|^{p+1} + \big(\beta - \frac{n+2}{q+1}\big)|v|^{q+1}\Big) dx dt, \end{aligned} \tag{3.13}$$

*where $n_\Omega$ is the outward pointing normal on $\partial\Omega$.*

*Proof of Lemma 3.2.* The first step is to integrate (2.8) over $\Omega\times[-T,T]$, $0<T<\infty$. This gives

$$\begin{aligned}\int_{\Omega\times[-T,T]}\operatorname{div}\mathbf{P} &= \oint_{\partial(\Omega\times[-T,T])}(\mathbf{P},n)\\ &= \int_{[-T,T]}\int_{\Omega}\Big(\big(\alpha-\frac{n+2}{p+1}\big)|u|^{p+1}+\big(\beta-\frac{n+2}{q+1}\big)|v|^{q+1}\Big),\end{aligned}\tag{3.14}$$

where $n$ is the outward pointing normal on $\partial(\Omega\times[-T,T])$. Due to Lemma 3.1 one can take the limit as $T\to\infty$ on the right hand side of (3.14). The next step will be to evaluate the left hand side of (3.14) and take the limit as $T$ goes to $\infty$. The boundary of $\Omega\times[-T,T]$ splits in three different parts:

$$\partial(\Omega\times[-T,T])=\partial\Omega\times[-T,T]\cup\Omega\times\{-T\}\cup\Omega\times\{T\},$$

and we determine the left hand side of (3.14) on the three consecutive parts, using the boundary conditions (1.3). The normal $n$ on these three parts is given by $\binom{n_\Omega}{0}$, $\binom{0}{-1}$ and $\binom{0}{1}$ respectively. This yields

$$\begin{aligned}\oint_{\partial(\Omega\times[-T,T])}(\mathbf{P},n) &= \int_{-T}^{T}\oint_{\partial\Omega}(\mathbf{P},n)+\Big(\int_\Omega(\mathbf{P},n)\Big)(T)+\Big(\int_\Omega(\mathbf{P},n)\Big)(-T)\\ &= -\int_{-T}^{T}\oint_{\partial\Omega}\Big[\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x,n_\Omega)-2tu_t(\nabla v\cdot n_\Omega)-2tv_t(\nabla u\cdot n_\Omega)\Big]\\ &\quad+2TH(u,v)(T)+2TH(u,v)(-T)\\ &\quad-\Big(\int_\Omega(x\cdot\nabla u)v\Big)(T)+\Big(\int_\Omega(x\cdot\nabla u)v\Big)(-T)\\ &\quad-\alpha\Big(\int_\Omega uv\Big)(T)+\alpha\Big(\int_\Omega uv\Big)(-T).\end{aligned}$$

The Hamiltonian $H(u,v)=\int_\Omega\mathcal{J}(u,v)$ is identically equal to zero for all $(x,t)\in\overline{\Omega}\times\mathbf{R}$, and $u_t$ and $v_t$ are zero at $\partial\Omega$ for all $t\in[-T,T]$, which yields

$$\begin{aligned}\oint_{\partial(\Omega\times[-T,T])}(\mathbf{P},n) &= -\int_{-T}^{T}\oint_{\partial\Omega}\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x,n_\Omega)+\alpha\Big(\int_\Omega uv\Big)(-T)-\alpha\Big(\int_\Omega uv\Big)(T)\\ &\quad+\Big(\int_\Omega(x,\nabla u)v\Big)(-T)-\Big(\int_\Omega(x,\nabla u)v\Big)(T).\end{aligned}\tag{3.15}$$

We have

$$\int_\Omega uv=\int_\Omega u(t)v(t)\le\Big|\int_\Omega u(t)v(t)\Big|\le C\|u(t)\|_{L^\infty}\|v(t)\|_{L^\infty},$$

and

$$\int_\Omega(x\cdot\nabla u)v\le\Big|\int_\Omega(x\cdot\nabla u)v\Big|\le C\|\nabla u\|_{L^\infty}\|u\|_{L^\infty}.$$

It follows from Lemma 3.1 and the above gradient estimates $(3.1)'$ and $(3.2)'$ that

$$\Big|\int_\Omega u(t)v(t)\Big|\le\begin{cases}Ce^{-(q+1)\gamma t}, & t\ge t_0,\\ Ce^{-(p+1)\gamma t}, & -t\ge t_0,\end{cases}$$

and

$$\Big|\int_\Omega (x\cdot\nabla u(t))v(t)\Big| \le \begin{cases} Ce^{-(q+1)\gamma t}, & t\ge t_0,\\ Ce^{-(p+1)\gamma t}, & -t\ge t_0.\end{cases}$$

This proves that $\left(\int_\Omega uv\right)(\pm T)\to 0$ and $\left(\int_\Omega (x\cdot\nabla u)v\right)(\pm T)\to 0$ as $T\to\infty$. Consequently the integral

$$-\int_{-T}^{T}\oint_{\partial\Omega}\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x,n_\Omega)$$

converges as $T\to\infty$, and the limit is denoted by

$$-\int_{\mathbf{R}}\oint_{\partial\Omega}\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x,n_\Omega).$$

Now we can take the limit as $T$ goes to $\infty$ in (3.14), which proves (3.13). □

We shall now use the fundamental identity (3.13) to prove Theorem 1.2. From the Hopf Boundary Point Lemma (e.g., see [10], [11]) for the operators $\frac{\partial}{\partial t}-\Delta$ and $-\frac{\partial}{\partial t}-\Delta$ it follows, for non-negative solutions $u,v$, that both

$$\frac{\partial u}{\partial n}<0,\quad \frac{\partial v}{\partial n}<0,\quad \forall(x,t)\in\partial\Omega\times\mathbf{R}. \tag{3.16}$$

Due to (3.16) the left hand side of (3.13) is strictly negative. On the other hand, if $p$ and $q$ satisfy (1.6) it follows that the right hand side of (3.13) is non-negative, which is a contradiction, unless $u=v\equiv 0$ in $\Omega\times\mathbf{R}$. This concludes the proof of Theorem 1.2.

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