HOPFIAN AND CO-HOPFIAN G-CW-COMPLEXES

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Abstract. We determine conditions for a G-CW-complex to be a Hopfian or a co-Hopfian object in the G-homotopy category of G-path-connected G-CW-complexes with base points.

1. Introduction

The notion of a Hopfian and a co-Hopfian object of a category is fairly well known. An object $X$ of a category $\mathcal{C}$ is called Hopfian (respectively, co-Hopfian) if every self-epimorphism (respectively, self-monomorphism) $f : X \to X$ is an equivalence; this notion plainly makes sense in any category, since epimorphisms and monomorphisms are categorically defined. It is interesting to recognize Hopfian and co-Hopfian objects in a specific category. Several results are known in this direction, [1], [5], [6], [7], [8], [9], [10], [11]. In [5] and [9], the authors studied Hopfian and co-Hopfian objects of $\mathcal{H}$, the homotopy category of pointed path-connected CW-complexes.

Let $G$ be a discrete group and $G\mathcal{H}$ denote the $G$-homotopy category of $G$-path-connected $G$-CW-complexes with base points (base points are $G$-fixed). In this paper we determine conditions for an object of $G\mathcal{H}$ to be a Hopfian or co-Hopfian object of $G\mathcal{H}$. Our results extend the results of [9] to the category $G\mathcal{H}$. Since $G$ is discrete, every object of $G\mathcal{H}$ is also an object of $\mathcal{H}$. We provide examples to show that the Hopficity and the co-Hopficity of an object $X$ of $G\mathcal{H}$ are independent of the Hopficity and the co-Hopficity of $X$ when considered as an object of $\mathcal{H}$.

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2. Hopfian objects

Let $O_G$ denote the category of canonical orbits. More precisely, objects of $O_G$ are homogeneous spaces $G/H$, $H$ a subgroup of $G$, and a morphism $\hat{g} : G/H \to G/K$ of $O_G$ is given by a subconjugacy relation $g^{-1}Hg \subset K$ (cf. [2]).

An abelian $O_G$-group is a contravariant functor from the category $O_G$ to the category $\mathbb{A}b$ of abelian groups. Such objects, along with obvious morphisms (natural transformations) between them form an abelian category $\mathcal{C}_G$. We shall denote the zero object in the abelian category $\mathcal{C}_G$ by $\mathbb{0} : O_G \to \mathbb{A}b$, $G/H \mapsto 0$, the trivial group.
If $X$ is an object of $GH$, then for every $i \geq 0$, we have an abelian $O_G$-group $H_iX : O_G \longrightarrow \text{Ab}$, defined by $H_iX(G/H) = H_i(X^H)$, the $i$-th integral homology group of $X^H$, $X^H$ being the $H$-fixed point set of $X$, for every object $G/H$ of $O_G$, and $H_iX(\hat{g}) = H_i(\hat{g}) : H_i(X^K) \longrightarrow H_i(X^H)$ for every morphism $\hat{g} : G/H \longrightarrow G/K$ of $O_G$, where $g : X^K \longrightarrow X^H$ is induced by the action of $G$ on $X$. Similarly, we have $O_G$-groups $\pi_*X$, $\pi^*_X$ need not be abelian. A morphism $f : X \longrightarrow Y$ of $GH$ induces a natural transformation $f_* : H_nX \longrightarrow H_nY$, where $f_*(G/H) = H_n(f^H) : H_n(X^H) \longrightarrow H_n(Y^H)$, $n \geq 0$.

We have the following easy lemma.

**Lemma 2.1.** A morphism $\eta : T \longrightarrow S$ in $C_G$ is an epimorphism (respectively, monomorphism) in $\text{Ab}$ if and only if $\eta(G/H) : T(G/H) \longrightarrow S(G/H)$ is an epimorphism (respectively, monomorphism) in $\text{Ab}$ for every object $G/H$ of $O_G$. \hfill $\square$

**Remark 2.2.** If $C'_G$ is the category of $O_G$-groups, then a morphism $\eta : T \longrightarrow S$ in $C'_G$ is a monomorphism if and only if $\eta(G/H)$ is a monomorphism in the category $\mathcal{G}$ of groups. If a morphism $\eta : T \longrightarrow S$ satisfies that $\eta(G/H)$ is onto for every object $G/H$ of $O_G$, then $\eta$ is an epimorphism in $C'_G$.

It follows immediately from the above discussion that:

**Proposition 2.3.** If an object $T$ in $C_G$ satisfies the condition that $T(G/H)$ is Hopfian (respectively, co-Hopfian) in $\text{Ab}$ for every object $G/H$ of $O_G$, then $T$ is a Hopfian (respectively, co-Hopfian) object in $C_G$. \hfill $\square$

Since the Hopfian and co-Hopfian objects in $\text{Ab}$ are by now well studied (cf. [1]), the above result gives an idea about the Hopfian and co-Hopfian objects in $C_G$.

**Definition 2.4.** A morphism $f : X \longrightarrow Y$ in $GH$ is a weak $G$-homotopy equivalence if $f_* : H_nX \longrightarrow H_nY$ is an isomorphism for every $n \geq 0$.

Note that if a morphism $f : X \longrightarrow Y$ of $GH$ is such that $f_*(G/H) : \pi_n(X^H) \longrightarrow \pi_n(Y^H)$ is an isomorphism for every $n \geq 0$, then $f$ is a $G$-homotopy equivalence [3].

**Proposition 2.5.** Let $f : X \longrightarrow Y$ be an epimorphism in $GH$. Then $f_* : H_kX \longrightarrow H_kY$ is an epimorphism in $C_G$ for all $k \geq 0$.

**Proof.** We may without loss of generality assume (by replacing $Y$ by the equivariant mapping cylinder of $f$) that $f$ is an inclusion. Then consider the maps $\pi : Y \longrightarrow Y/X$ and $c : Y \longrightarrow Y/X$, where $\pi$ is the quotient and $c$ is the constant $G$-map. Then $\pi \circ f = c \circ f$. Since $f$ is an epimorphism, it follows that $\pi$ is $G$-homotopic to $c$. Now for every $H \subset G$, it follows from the exact homology sequence

$$
\cdots \longrightarrow H_k(X^H) \longrightarrow H_k(Y^H) \longrightarrow H_k((Y/X)^H) = H_k(Y^H/X^H) \longrightarrow \cdots
$$

that $f_*^H : H_k(X^H) \longrightarrow H_k(Y^H)$ is an epimorphism in $\text{Ab}$ for every $k \geq 1$. The result follows from Lemma 2.1. \hfill $\square$

**Remark 2.6.** Note that for any morphism $f : X \longrightarrow Y$ in $GH$, $f_* : H_0X \longrightarrow H_0Y$ is an isomorphism. This follows from the fact that $H_0(X^H)$ is generated by the homology class of the base point $x_0 \in X^G$ in $H_0(X^H)$, and $f$ being a morphism in $GH$, $(f^H)$, maps the generator of $H_0(X^H)$ onto the generator of $H_0(Y^H)$.

**Theorem 2.7.** Let $f : X \longrightarrow X$ be a self-epimorphism in $GH$. If $H_nX$, $n \geq 1$ are Hopfian objects in $C_G$, then $f$ is a weak $G$-homology equivalence.
Proof. By the Remark 2.6 $f_* : \mathcal{H}_G X \rightarrow \mathcal{H}_G X$ is an isomorphism. Since $f$ is an epimorphism, Proposition 2.5 implies that $f_* : \mathcal{H}_G X \rightarrow \mathcal{H}_G X$ is an epimorphism for every $n \geq 1$. The result now follows as $\mathcal{H}_G X$, $n \geq 1$ are Hopfian objects in $\mathcal{C}_G$.

Recall from [4] the following definition.

Definition 2.8. A $G$-space $X$ is nilpotent if each $\mathcal{H}_G X$, $n \geq 1$ is nilpotent as an $O_G$-module over $\mathcal{H}_X$, that is, there are $O_G$-submodules

$$\{0\} = \mathcal{H}_n, 0 X \subset \mathcal{H}_n, 1 X \subset \cdots \subset \mathcal{H}_n, r_n X = \mathcal{H}_n X$$

such that the subquotients $A_{n,j} = \mathcal{H}_n, j+1 X / \mathcal{H}_n, j X$ are abelian with trivial $\mathcal{H}_j X$-action.

This is equivalent to saying each $X^H$ is nilpotent in the usual sense with a uniform bound on the order of nilpotence in each dimension (of course, this last condition is vacuous if $G$ is finite).

Corollary 2.9. Let an object $X$ of $\mathcal{G}H$ be nilpotent as a $G$-space, and $\mathcal{H}_n X$, $n \geq 1$ are Hopfian in $\mathcal{C}_G$, then $X$ is Hopfian in $\mathcal{G}H$.

Proof. Let $f : X \rightarrow X$ be a self-epimorphism in $\mathcal{G}H$. Then by Theorem 2.7 $f$ is a weak $G$-homotopy equivalence. But $X^H$ being nilpotent for every $H \subset G$, it follows that $f^H : X^H \rightarrow X^H$ is a homotopy equivalence and hence $f : X \rightarrow X$ is a $G$-homotopy equivalence. This completes the proof.

It may be noted that Corollary 1.1 of [9] follows from Corollary 2.9 by taking $G$ to be the trivial group.

Let $\lambda : O_G \rightarrow \mathcal{G}$ be an $O_G$-group, and $K(\lambda, 1)$ denote the equivariant Eilenberg-Mac Lane complex of the type $(\lambda, 1)$ [4]. It may be remarked that for any $O_G$-group $\lambda : O_G \rightarrow \mathcal{G}F$, $K(\lambda, n)$ is the classifying space for the Bredon cohomology with coefficient $\lambda$ [2].

Proposition 2.10. For any object $X$ of $\mathcal{G}H$ and $O_G$-group $\lambda : O_G \rightarrow \mathcal{G}$ there is an adjunction equivalence $[X, K(\lambda, 1)]_G \leftrightarrow \text{Hom}(\mathcal{H}_1 X, \lambda)$.

Proof. If $f : X \rightarrow K(\lambda, 1)$ represents an element of $[X, K(\lambda, 1)]_G$, then the corresponding natural transformation in $\text{Hom}(\mathcal{H}_1 X, \lambda)$ is given by $f_* : \mathcal{H}_1 X \rightarrow \lambda$ (note that $\mathcal{H}_1 K(\lambda, 1) = \lambda$). Conversely, a natural transformation $T : \mathcal{H}_1 X \rightarrow \lambda$ induces a $G$-map $T_* : K(\mathcal{H}_1 X, 1) \rightarrow K(\lambda, 1)$ (cf. [4]). Note that $X$ can be regarded as a $G$-subcomplex of $K(\mathcal{H}_1 X, 1)$, for we may obtain $K(\mathcal{H}_1 X, 1)$ from $X$ by attaching suitable equivariant cells to $X$ to kill the higher homotopy groups of the fixed point sets of $X$. The class represented by $T_* / X$ in $[X, K(\lambda, 1)]_G$ is then the element which corresponds to $T$.

It follows immediately from Proposition 2.10 that :

Proposition 2.11. If $f : X \rightarrow Y$ is an epimorphism in $\mathcal{G}H$, then $f_* : \mathcal{H}_1 X \rightarrow \mathcal{H}_1 Y$ is an epimorphism in $\mathcal{C}_G$.

Corollary 2.12. If $\lambda : O_G \rightarrow \mathcal{G}$ is Hopfian in $\mathcal{C}_G$, then $K(\lambda, 1)$ is a Hopfian object in $\mathcal{G}H$.

Corollary 2.13. If $X$ is $G-\langle n-1 \rangle$-connected, $n > 1$, (that is, each $X^H$ is $(n-1)$-connected) and $f : X \rightarrow X$ is an epimorphism in $\mathcal{G}H$, then $f_* : \mathcal{H}_n X \rightarrow \mathcal{H}_n X$ is an epimorphism.
Proof. Note that since $X$ is $G$-$(n-1)$-connected, the natural transformation $\pi_n X \to H_n X$ given by the Hurewicz homomorphism is actually an isomorphism. The result now follows from Proposition 2.5.

Thus in view of Remark 2.2, it follows that if $\lambda : O_G \to G$ is such that $\lambda(G/H)$ is a Hopfian group for every $H \subset G$, then $K(\lambda, 1)$ is a Hopfian object in $G\mathcal{H}$. In fact, if $\lambda : O_G \to Ab$ is a Hopfian object in $C_G$, then $K(\lambda, n)$ is Hopfian for every integer $n > 1$. To see this, we first need to prove the following result.

**Proposition 2.14.** If $X$ is $G$-$(n-1)$-connected, $n > 1$, then there is an adjunction equivalence $[X, K(\lambda, n)]_G \leftrightarrow Hom(\pi_n X, \lambda)$, for any $\lambda : O_G \to Ab$.

Proof. Recall from [2] that there exists a spectral sequence whose $E_2$ term is $E^{p,q}_2 = \text{Ext}^p(H^q(X), \lambda) \Rightarrow H^{p+q}(X; \lambda)$, here $H^{p,q}(X; \lambda)$ is the Bredon cohomology group of $X$ with coefficient $\lambda$. There is an edge homomorphism $H^{p,q}_G(X; \lambda) \to Hom(H^p(X; \lambda)$ of the above spectral sequence, which is an isomorphism if $H^q X$ is projective for $q < n$. Now, since $X$ is $G$-$(n-1)$-connected, $H_q X = 0$ for $0 < q < n$ and $H_n X \cong \pi_n X$, where $0 : O_G \to Ab$ is the zero object in the category $C_G$. Moreover, since $X$ is $G$-path-connected, $H_0 X(G/H) = \mathbb{Z}(x^0)$, where $x^0$ is the base point and $(x^0)$ is the homology class of $x^0$ and $H_0 X(\hat{g}) = id$. The result now follows from the fact that if $B$ is projective in $Ab$, then $B$ is projective in $C_G$, where $B$ is defined by $B(G/H) = B$ for every $H \subset G$ and $B(\hat{g}) = id$, for every morphism $\hat{g} : G/H \to G/K$ of $O_G$.

**Corollary 2.15.** If $\lambda : O_G \to Ab$ is Hopfian in $C_G$, then $K(\lambda, n)$ is Hopfian in $G\mathcal{H}$.

**Example 2.16.** Let $X$ be a $G$-connected finite $G$-CW-complex (that is, $X$ has a finite number of equivariant cells) which has one $G$-fixed 0-cell and no 1-cell. Since $H_i(X^H)$ is finitely generated abelian for every $H \subset G$, by Lemma 2.1 $H_n X$ is Hopfian. Moreover it is clear that $X$ is $G$-simply-connected and hence nilpotent. Thus by Corollary 2.9, $X$ is Hopfian in $G\mathcal{H}$.

**Example 2.17.** Consider the real $4k$-dimensional Euclidean space $\mathbb{R}^{4k}$ as the quaternion $k$-space $\mathbb{H}^k$. Let $\tau$ be a quaternion of norm one and order $p$, an odd prime. We can take $\tau = e^{\pi i/p}$. Define an action of $\mathbb{Z}_p$ on $\mathbb{R}^{4k} \cong \mathbb{H}^k$ by

$$\tau(a_1, a_2, \ldots, a_k) = (\tau a_1 \tau^{-1}, \tau a_2 \tau^{-1}, \ldots, \tau a_k \tau^{-1}),$$

for any $k$-tuple of quaternions $(a_1, a_2, \ldots, a_k)$. Since this action is norm preserving, there results a $\mathbb{Z}_p$-action on the $(4k - 1)$-sphere $S^{4k-1}$. The fixed point sets are $S^{4k-1}$ and $(S^{4k-1})^{\mathbb{Z}_p}$. To determine $(S^{4k-1})^{\mathbb{Z}_p}$ we proceed as follows. Let $(a_1, a_2, \ldots, a_k) \in (S^{4k-1})^{\mathbb{Z}_p}$, where $a_r = a^1_r + a^2_r i + a^3_r j + a^4_r k = A^1_r + A^2_r j$, and $A^1_r = a^1_r + a^2_r i$, $A^2_r = a^2_r + a^3_r i$, $r = 1, 2, \ldots, k$. We must have $\tau a_r = a_r \tau$. Now $\tau a_r = \tau A^1_r + \tau A^2_r$, whereas $a_r = (A^1_r + A^2_r) \tau = A^1_r \tau + A^2_r \tau$. Thus we must have $\tau A^1_r = \mathbb{T} A^1_r$ or $\tau = \mathbb{T} A^2_r$. For $r = 1, 2, \ldots, k$, we must have $\tau A^1_r = \mathbb{T} A^2_r$ or $\tau = \mathbb{T} A^2_r$. Therefore, $A^2_r = 0$, as $\tau \neq \mathbb{T}$. Therefore $(a_1, a_2, \ldots, a_k) \in \mathbb{C}^k$, with $||(a_1, a_2, \ldots, a_k)|| = 1$. Thus $(S^{4k-1})^{\mathbb{Z}_p} = S^{2k-1}$. Now $S^{4k-1}$ is a smooth compact $\mathbb{Z}_p$-manifold, it admits a structure of a finite $\mathbb{Z}_p$-CW-complex on which $\mathbb{Z}_p$-path-connected and has a base point. Moreover, note that the fixed point sets $S^{4k-1}$ and $S^{2k-1}$ being simply-connected, are nilpotent. It is now easy to check that all the conditions of Corollary 2.9 are satisfied and hence it is a Hopfian object in $G\mathcal{H}$ where $G = \mathbb{Z}_p$. 

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3. Co-Hopfian objects

In this section we obtain conditions for an object $X$ of $\mathcal{H}$ to be a co-Hopfian object. The following proposition is a straightforward consequence of Proposition 2.10.

**Proposition 3.1.** $f : S \to T$ is a monomorphism in $\mathcal{C}_G'$ if and only if the induced map $f_* : K(S, 1) \to K(T, 1)$ is a monomorphism in $\mathcal{H}$. □

**Corollary 3.2.** For an object $\lambda : O_G \to \mathcal{G}$ in $\mathcal{C}_G'$, $K(\lambda, 1)$ is co-Hopfian if and only if $\lambda$ is co-Hopfian. □

As before, we may obtain from Proposition 2.14 that:

**Corollary 3.3.** For any $\lambda : O_G \to \text{Ab} \in \mathcal{C}_G$ and $n > 1$, $K(\lambda, n)$ is co-Hopfian in $\mathcal{H}$ if and only if $\lambda$ is co-Hopfian in $\mathcal{C}_G$. □

**Definition 3.4.** For an object $X$ of $\mathcal{H}$, we say $X$ is finitely generated if $\pi_i X(G/H) = \pi_i (X^H)$ is finitely generated for every $H \subset G$. $X$ will be called $G$-homotopically finite type if $X$ is finitely generated for all $i \geq 2$.

**Theorem 3.5.** Suppose an object $X$ of $\mathcal{H}$ is $G$-homotopically finite type and such that $\pi_i (X^H)$ is a co-Hopfian group and the inclusion $X^H \subset X$ is a monomorphism in $\mathcal{H}$ for every $H \subset G$. Then $X$ is a co-Hopfian object in $\mathcal{H}$.

**Proof.** Let $f : X \to X$ be a self-monomorphism in $\mathcal{H}$. We show that under the given hypothesis $f^H : X^H \to X^H$ is a monomorphism in $\mathcal{H}$ for every $H \subset G$. Since $\pi_i (X^H)$ is finitely generated for all $i \geq 2$ and $\pi_1 (X^H)$ is co-Hopfian, it will follow from Theorem 7 and Corollary 2 of [5] that $f^H : X^H \to X^H$ is a homotopy equivalence. Hence $f$ is a $G$-homotopy equivalence.

First we show that $f = f^{(e)} : X = X^{(e)} \to X^{(e)} = X$ is a monomorphism in $\mathcal{H}$. We assume that the base point $x^0 \in X^G$ is a $G$-fixed 0-cell in $X$. Let $\alpha, \beta : Y \to X$ be morphisms in $\mathcal{H}$ such that $f \circ \alpha \simeq f \circ \beta$. Let $F : Y \times I \to X$ be the homotopy $f \circ \alpha \simeq f \circ \beta$. Consider $Y \times G$ as a $G$-space, where the action of $G$ is given by $g(y, h) = (y, gh)$, for all $g \in G, h \in G, y \in Y$. Clearly, the above action is free. Let $y^0$ be the base point of $Y$, which is a 0-cell of $Y$. Define $\overline{\pi} : Y \times G \to X$ by $\overline{\pi}(y, e) = \alpha(y)$ and $\overline{\pi}(y, g) = g \alpha(y)$. Then $\overline{\pi}$ is a $G$-map. Note that $\overline{\pi}(y^0, g) = x^0$ for all $g \in G$. Let $Y_G$ be the space obtained from $Y \times G$ by identifying all points $(y^0, g), g \in G$. Then $Y_G$ is a $G$-complex having a natural base point, which is a $G$-fixed 0-cell and is clearly an object of $\mathcal{H}$. The map $\overline{\pi}$ induces a $G$-map $\overline{\alpha} : Y_G \to X$ which is base point preserving. Similarly, we have $\overline{\beta} : Y_G \to X$. The homotopy $F : Y \times I \to X$ gives rise to a $G$-homotopy $\overline{F} : Y \times G \times I \to X$, between $f \circ \overline{\pi}$ and $f \circ \overline{\beta}$, by setting $\overline{F}(y, e, t) = F(y, t)$ and $\overline{F}(y, g, t) = g F(y, t)$ for all $g \in G, t \in I$. Since the homotopy $F$ is base point preserving, $\overline{F}$ induces a $G$-homotopy $\overline{F} : Y_G \times I \to X$, such that $\overline{F}$ is a $G$-homotopy between $f \circ \overline{\alpha}$ and $f \circ \overline{\beta}$. Since $f$ is a monomorphism in $\mathcal{H}$, $\overline{\alpha}$ is $G$-homotopic to $\overline{\beta}$. Let $\overline{F}_1 : Y_G \times I \to X$ be a $G$-homotopy between them. Let $i : Y \to Y_G$ be the imbedding $y \mapsto [y, e]$. Let $F_1 : Y \times I \to X$ be the composition of $i \times id : Y \times I \to Y_G \times I$ and $\overline{F}_1$. Then it is easy to see that $F_1 : \alpha \simeq \beta$. Thus $f^{(e)} : X^{(e)} \to X^{(e)}$ is a monomorphism in $\mathcal{H}$.

Next, let $H \subset G$. Let $\alpha, \beta : Y \to X^H$ be any two morphisms in $\mathcal{H}$ such that $f^H \circ \alpha \simeq f^H \circ \beta$. Let $i$ denote the inclusion $X^H \subset X$. Then, $i \circ f^H \circ \alpha \simeq i \circ f^H \circ \beta$. This implies $f \circ i \circ \alpha \simeq f \circ i \circ \beta$, since $f$ being a $G$-map $i \circ f^H = f \circ i$. Since $f^{(e)}$
Example 4.1. Therefore $X$ is a monomorphism in $\mathcal{H}$, we conclude $i \circ \alpha \simeq i \circ \beta$. Now since $i : X^H \subset X$ is a monomorphism, it follows that $\alpha \simeq \beta$. Therefore, $f^H$ is a monomorphism in $\mathcal{H}$. This completes the proof of the theorem.

As an immediate corollary we get

Corollary 3.6. Suppose $X$ is an object of $G\mathcal{H}$ such that the action of $G$ is semifree and $X^G = \{x^0\}$, $x^0$ is the $G$-fixed 0-cell. Moreover, suppose that $\pi_i(X)$ is finitely generated for $i \geq 2$ and $\pi_1(X)$ is a co-Hopfian group. Then $X$ is a co-Hopfian object in $G\mathcal{H}$.

Example 3.7. Let $n \geq 2$ and $X = S^n \vee S^n$. Then $X$ has a $\mathbb{Z}_2$-CW-complex structure as described below. It has one 0-cell of the type $\mathbb{Z}_2/\{e\}$, and one equivariant $n$-cell of the type $\mathbb{Z}_2/\{e\}$, where $e$ denotes the identity element of $\mathbb{Z}_2$. This action is given by “switching coordinates”, regarding the wedge as a subspace of the Cartesian product $S^n \times S^n$. Since $X$ is a 1-connected finite complex, $\pi_q(X)$ is finitely generated. Moreover, $\pi_1(X) = \{0\}$. Hence it follows from Corollary 3.6 that $X$ is co-Hopfian in $G\mathcal{H}$ where $G = \mathbb{Z}_2$.

4. $G\mathcal{H}$ versus $\mathcal{H}$

Recall that if $G$ is discrete and $X$ a $G$-CW-complex, then $X$ is in a canonical way a CW-complex (cf. [3], p. 102). Thus if $X$ is an object of $G\mathcal{H}$, then $X$ can also be regarded as an object of $\mathcal{H}$. We shall show by the following examples that an object $X$ of $G\mathcal{H}$ can be Hopfian (respectively, co-Hopfian) in $G\mathcal{H}$ without being Hopfian (respectively, co-Hopfian) in $\mathcal{H}$, and vice versa.

Example 4.1. Let $G = \mathbb{Z}_2$. Define an $O_G$-group $\lambda : O_G \rightarrow \mathbb{A}b$ as follows. $\lambda(G/G) = \mathbb{Z}$, $\lambda(G/\{e\}) = \{0\}$, the trivial group, and $\lambda(G/\{e\}) \rightarrow G/G : \mathbb{Z} \rightarrow \{0\}$ is the obvious homomorphism. Let $X = K(\lambda, 1)$. Then $X$ is co-Hopfian in $\mathcal{H}$, but not co-Hopfian in $G\mathcal{H}$.

To see this, note that $X = X^{(e)} = K(\lambda(G/\{e\}), 1)$. Hence $X$ is contractible. Therefore $X$ is co-Hopfian in $\mathcal{H}$. Next, note that $\lambda$ is not co-Hopfian in $C_G$. For, $\eta : \lambda \rightarrow \lambda$ defined by $\eta(G/G) : x \mapsto 2x$, $\eta(G/\{e\}) = id_{\{0\}}$ is a monomorphism, but not an isomorphism in $C_G$ by Lemma 2.1. It follows from Corollary 3.2 that $X$ is not co-Hopfian in $G\mathcal{H}$.

Let $G = \mathbb{Z}$, and $H_n$ denote the subgroup $2^n\mathbb{Z}$, $n \geq 0$. If $H$ is a subgroup of $G$, $H \neq H_n$ for all $n$, then $H = k\mathbb{Z}$, where $k = 2^n \ell$, $\ell$ odd , $\ell$ is not 1 or $-1$ and $n_i \geq 0$. Clearly $k\mathbb{Z} \subset H_n$, and there is no subconjugacy relation of the type $H_m \subset k\mathbb{Z}$. Also note that $H_{n+1} \subset H_n$ for all $n$. We define an $O_G$-group $\lambda : O_G \rightarrow \mathbb{A}b$ as follows. Let $Q^\infty$ denote the direct sum $\bigoplus_i Q e_i$ of countable copies of $Q$ with basis $\{e_1, e_2, \ldots, e_n, \ldots\}$. Thus $Q^\infty$ is a vector space over $Q$. Clearly, $Q^\infty$ is neither Hopfian nor co-Hopfian in $\mathbb{A}b$. Let $Q^n = \bigoplus_{i=1}^n Q e_i$. Note that every group homomorphism $Q^n \rightarrow Q^n$ is actually a $Q$-linear homomorphism $Q^n \rightarrow Q^n$. Then it is easy to see that $Q^n$ is both Hopfian and co-Hopfian in $\mathbb{A}b$.

Set $\lambda(G/\{e\}) = Q^\infty$, $\lambda(G/H_n) = Q^n$ for all $n \geq 0$. If $H = k\mathbb{Z}$, $k = 2^n \ell$, $\ell$ odd and not equal to 1 or $-1$, $n_i \geq 0$, then we set $\lambda(G/k\mathbb{Z}) = Q^{n_i}$. Here, $Q^0 = \{0\}$, the trivial group. For every subgroup relation $H_{n+1} \subset H_n$, let

$$
\lambda(G/H_{n+1} \rightarrow G/H_n) : Q^n \rightarrow Q^{n+1}
$$

be the standard inclusion. For $k\mathbb{Z} \subset H_n$, $k = 2^n \ell$, $\ell$ odd and not equal to 1 or $-1$, $n_i \geq 0$, let $\lambda(G/k\mathbb{Z} \rightarrow G/H_n) : Q^{n_i} \rightarrow Q^{n_i}$ be the identity. Again, for the

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inclusions \( \{e\} \subset H_n \) and \( \{e\} \subset k\mathbb{Z}, \) \( k = 2^m \ell, \) \( \ell \) odd and not equal to 1 or \(-1, \) \( n_i \geq 0, \) we set \( \lambda(G/\{e\}) \to G/H_n : Q^n \to Q^\infty \) and \( \lambda(G/\{e\}) \to G/k\mathbb{Z} : Q^n_k \to Q^\infty \) to be the obvious inclusions. Then it is easy to see that \( \lambda \) is a contravariant functor from \( O_G \) to \( Ab. \)

Example 4.2. Let \( \lambda \) be as above and \( X = K(\lambda, 1) \). Then \( X \) is co-Hopfian in \( G\mathcal{H}, \) but not co-Hopfian in \( \mathcal{H}. \)

Since \( X = X^{\{e\}} = K(\lambda(G/\{e\}), 1) \) and \( Q^\infty \) is not co-Hopfian, it follows that \( X \) is not co-Hopfian in \( \mathcal{H}. \) To show that \( X \) is co-Hopfian in \( G\mathcal{H}, \) by Corollary 3.2, it is enough to show that \( \lambda \) is co-Hopfian. Let \( \eta : \lambda \to \lambda \) be a monomorphism. Then, by Lemma 2.1, \( \eta(G/H) : \lambda(G/H) \to \lambda(G/H) \) is a monomorphism for every subgroup \( H \) of \( G. \) By construction of \( \lambda, \) it is clear that \( \eta(G/H) \) is an isomorphism for every subgroup \( H \neq \{e\}. \) We shall show that \( \eta(G/\{e\}) \) is also an isomorphism. Let \( x \in Q^\infty. \) Then we can find \( n \) such that \( x \in Q^n. \) Since \( \eta(G/H_n) \) is an isomorphism, \( x \) lies in the image of \( \eta(G/H_n). \) By naturality of \( \eta \) we have

\[
\eta(G/\{e\})\lambda(G/\{e\}) \to G/H_n = \lambda(G/\{e\}) \to G/H_n \eta(G/H_n).
\]

It follows from Lemma 2.1 that \( \eta \) is an isomorphism. Thus \( \lambda \) is co-Hopfian.

Example 4.3. Let \( G = \mathbb{Z}, \) and \( \lambda : O_G \to Ab \) be as in Example 4.2. Let \( X = K(\lambda, 1). \) Then \( X \) is Hopfian in \( G\mathcal{H}, \) but not Hopfian in \( \mathcal{H}. \)

By an argument similar to the previous case, one can show that every epimorphism \( \eta : \lambda \to \lambda \) is an isomorphism. Thus \( \lambda \) is Hopfian. It follows from Corollary 2.12 that \( X \) is Hopfian in \( G\mathcal{H}. \) To show that \( X \) is not Hopfian in \( \mathcal{H}, \) it is enough to produce a self-epimorphism of \( X \) which is not an equivalence. Since \( Q^\infty \) is not Hopfian, we have an epimorphism \( f : Q^\infty \to Q^\infty \) which is not an isomorphism. Let \( F : X = K(Q^\infty, 1) \to K(Q^\infty, 1) = X \) be the map induced by \( f. \) Clearly, \( F \) is not an equivalence as \( \pi_1(F) = f \) is not an isomorphism. We claim that \( F \) is an epimorphism. Let \( \alpha, \beta : X \to Y \) be base point preserving maps such that \( \alpha \circ F \simeq \beta \circ F. \) Since \( f : Q^\infty \to Q^\infty \) is surjective, there exists a homomorphism \( s : Q^\infty \to Q^\infty \) such that \( f \circ s = id. \) Let \( S : X \to X \) be the map induced by \( s. \) Then \( F \circ S \simeq id_X. \) This implies \( \alpha \simeq \beta. \) Thus \( F \) is an epimorphism. Hence \( X \) is not Hopfian in \( \mathcal{H}. \)

Example 4.4. Let \( G = \mathbb{Z}_2, \) and \( \lambda : O_G \to Ab \) be the \( O_G \)-group defined as follows: \( \lambda(G/G) = Q^\infty, \lambda(G/\{e\}) = \{0\}, \) and \( \lambda(G/\{e\}) \to G/G : Q^\infty \to \{0\} \) is the obvious homomorphism. Then \( X = K(\lambda, 1) \) is Hopfian in \( \mathcal{H}, \) but not Hopfian in \( G\mathcal{H}. \)

Clearly \( X \) is Hopfian in \( \mathcal{H}, \) as \( X \) is contractible. To see that \( X \) is not Hopfian in \( G\mathcal{H}, \) it is enough to find an epimorphism in \( G\mathcal{H} \) which is not an equivalence. Let \( \alpha : \lambda \to \lambda \) be the natural transformation defined as follows: \( \alpha(G/\{e\}) = id_{\{0\}} \) and \( \alpha(G/G) : Q^\infty \to Q^\infty \) any epimorphism which is not an isomorphism. Let \( \beta : Q^\infty \to Q^\infty \) be a homomorphism such that \( \alpha(G/G) \circ \beta = id. \) This defines a right inverse \( \beta : \lambda \to \lambda \) of \( \alpha, \) where \( \beta(G/G) = \beta \) and \( \beta(G/\{e\}) = id_{\{0\}}. \) Let \( T, S : X \to X \) be the \( G \)-maps induced by \( \alpha \) and \( \beta \) respectively. Then \( T \circ S \) is \( G \)-homotopic to \( id_X. \) Clearly, \( T \) is not a \( G \)-equivalence, as \( \pi_1(TG) = \alpha(G/G) \) is not an isomorphism. Now proceeding as in Example 4.3 one shows that \( T : X \to X \) is an epimorphism in \( G\mathcal{H}. \)
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