SEQUENTIAL TYPE KOROVKIN THEOREM ON $L^\infty$ FOR QC-TEST FUNCTIONS

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Abstract. Let $\{T_n\}$ be a sequence of bounded linear operators on $L^\infty$ such that $\|T_n\| \to 1$ and $\|T_n g - g\|_\infty \to 0$ for every $g \in QC$. It is proved that $\|T_n f - f\|_\infty \to 0$ for every $f \in L^\infty$.

1. Introduction

In 1953, Korovkin [8] (see also [9]) proved the following exciting approximation theorem: if $\{T_n\}$ is a sequence of positive linear operators on $C([0, 1])$ such that $\|T_n t^j - t^j\|_\infty \to 0$ for $j = 0, 1, 2$ as $n \to \infty$, then $\|T_n f - f\|_\infty \to 0$ for every $f \in C([0, 1])$. In 1968, Wulbert [14] proved that this theorem is also true if the condition “positivity” is replaced by the one “$\|T_n\| \to 1$”. Recently, there has been much research on this subject, Korovkin type approximation theorems; see the monograph by Altomare and Campiti [1].

Let $\Omega$ be a compact Hausdorff space and let $C(\Omega)$ be the space of complex valued continuous functions on $\Omega$. For a closed subset $E$ of $\Omega$ and $f \in C(\Omega)$, let $\|f\|_E = \sup \{|f(x)|; x \in E\}$. When $E = \Omega$, we write $\|f\|_\infty = \|f\|_{\Omega}$. Let $S$ be a closed subspace of $C(\Omega)$. We say that the sequential type Korovkin approximation theorem holds on $C(\Omega)$ if for every sequence of bounded linear operators $\{T_n\}$ on $C(\Omega)$ such that $\|T_n\| \to 1$ and $\|T_n g - g\|_\infty \to 0$ for every $g \in S$, then it follows that $\|T_n f - f\|_\infty \to 0$ for every $f \in C(\Omega)$. $S$ is called test functions. We can also consider the net type Korovkin approximation theorem by replacing the condition “a sequence $\{T_n\}$” by “a net $\{T_\alpha\}$”. The interesting problem is for which $S$ the sequential type Korovkin theorem holds on $C(\Omega)$. Takahasi [13] (see [14]) proved that the net type Korovkin theorem holds on $C(\Omega)$ for $S$ if and only if the Choquet boundary of $S$ coincides with $\Omega$. It is easy to see that if the net type Korovkin theorem holds for $S$ then the sequential type Korovkin theorem holds for $S$. In [6], the author, Takagi and Watanabe show that if $S$ is separable then the converse of the above fact is true. In [12], Scheffold gave the example of $S$ such that the sequential type Korovkin theorem holds but the net type Korovkin theorem does not hold. The given $S$ in [12] is the closed ideal of $C(\Omega)$ with some additional properties. As a completion of Scheffold’s result, in [7] the author, Takagi and Watanabe prove that the sequential type Korovkin theorem holds on $C(\Omega)$ for a closed ideal $S$ with $S = \{f \in C(\Omega); f = 0$ on $\Gamma\}$, where $\Gamma$ is a closed subset of $\Omega$, if...
and only if $\Gamma$ does not contain any quasi $G_\delta$-subsets of $\Omega$, where a closed subset $E$ is called quasi $G_\delta$ if there exists a sequence of open subsets $\{U_n\}_n$ of $\Omega$ such that $U_{n+1} \subset U_n$ and $E = \bigcap_{n=1}^\infty U_n$, where $U_n$ is the closure of $U_n$ in $\Omega$.

It seems very difficult to give a complete characterization of $C^*$-subalgebras $S$ of $C(\Omega)$ for which the sequential type Korovkin approximation theorem holds. In this paper, we study the sequential type Korovkin theorem on the unit circle.

Let $D$ be the open unit disk and $\partial D$ the unit circle. Let $H^\infty$ be the Banach algebra of boundary functions of bounded analytic functions on $D$. Then $H^\infty$ is the essential supremum norm closed subalgebra of $L^\infty$, the Banach algebra of bounded measurable functions on $\partial D$. We denote by $M(L^\infty)$ the maximal ideal space of $L^\infty$. We consider that $L^\infty = C(M(L^\infty))$. It is well known that $H^\infty + C$ is the closed subalgebra of $L^\infty$ [11], where $C$ is the space of continuous functions on $\partial D$.

Let

$$QC = (H^\infty + C) \cap (H^\infty + C) \quad \text{and} \quad QA = H^\infty \cap QC.$$  

Then $C \subset QC \subset L^\infty$ and the $C^*$-algebra $QC$ is studied by Sarason [10] extensively. The purpose of this paper is to prove that the sequential type Korovkin theorem holds on $L^\infty$ for test functions $QC$. Since $QC$ does not separate the points in $M(L^\infty)$, by Takahasi’s criterion the net type Korovkin theorem does not hold on $L^\infty$ for $QC$. Also we note that the sequential type Korovkin theorem does not hold on $L^\infty$ for $C$.

In the same way, we can prove that the sequential type Korovkin theorem holds on $H^\infty$ for test functions $QA$.

2. Preliminaries

Let $M(H^\infty)$ be the maximal ideal space of $H^\infty$. We identify a function in $H^\infty$ with its Gelfand transform on $M(H^\infty)$. Then we may consider that $M(L^\infty) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary of $H^\infty$. Also we can consider that $D \subset M(H^\infty)$, and by the corona theorem $D$ is dense in $M(H^\infty)$. The references [2] and [3] are nice for the spaces $H^\infty$ and $L^\infty$.

For a subset $E$ of $M(L^\infty)$, we denote by $\overline{E}$ the closure of $E$ in $M(L^\infty)$. The outstanding topological property of $M(L^\infty)$ is that if $U$ is an open subset of $M(L^\infty)$ then $\overline{U}$ is also open. For a point $x$ in $M(H^\infty)$, there exists a unique probability measure $\mu_x$ on $M(L^\infty)$ such that

$$\int_{M(L^\infty)} f \, d\mu_x = f(x) \quad \text{for every } f \in H^\infty.$$

We denote by $\operatorname{supp}\mu_x$ the closed support set of $\mu_x$. The following is a characterization of functions in $QC$.

**Lemma 1** ([10]). Let $f \in L^\infty$. Then $f \in QC$ if and only if $f$ is constant on $\operatorname{supp}\mu_x$ for every $x \in M(H^\infty) \setminus D$.

For a point $x$ in $M(L^\infty)$, let

$$Q_x = \{y \in M(L^\infty); f(y) = f(x) \quad \text{for every } f \in QC\}.$$

The set $Q_x$ is called the QC-level set associate with $x$. For a point $\zeta$ in $M(H^\infty) \setminus D$, there corresponds a QC-level set $Q_\zeta$ such that $\operatorname{supp}\mu_\zeta \subset Q_\zeta$. For a function $f$ in $L^\infty$, we denote by $N(f)$ the closure of

$$\bigcup \{\operatorname{supp} \mu_x; x \in M(H^\infty) \setminus D \text{ and } f|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x\}.$$
By the author [4, 5], the set \( N(f) \) was investigated extensively. In this paper, the \( N(f) \) plays the essential role.

**Lemma 2** ([5, Corollary 2.1]). Let \( f \in L^\infty \). Then \( N(f) = \cup \{Q_x; x \in N(f)\} \).

**Lemma 3** ([4, Corollary 7]). Let \( f_1, f_2 \in L^\infty \). Then \( N(f_1) \cup N(f_2) \) does not contain any \( G_k \)-subsets of \( M(L^\infty) \).

For \( f \in L^\infty \), let

\[
\tilde{N}(f) = N(f) \cup N(\tilde{f}).
\]

Then \( \tilde{N}(f) \) coincides with the closure of \( \cup\{\text{supp}_{\mu_x}; x \in M(H^\infty) \setminus D \text{ and } f|\text{supp}_{\mu_x} \text{ is not constant}\} \). If \( f \in H^\infty \), then \( \tilde{N}(f) = N(f) \). The following is a key to proving our theorem.

**Lemma 4.** Let \( f \in L^\infty \) and let \( \{V_n\}_n \) be a sequence of open and closed subsets of \( M(L^\infty) \) such that \( V_n \cap \tilde{N}(f) = \emptyset \) for every \( n \). Then there exist a subsequence \( \{n_j\}_j \) of positive integers and \( x_{n_j} \in V_{n_j} \) such that \( \{x_{n_j}\}_j \cap \tilde{N}(f) = \emptyset \).

**Proof.** Let

\[
W_n = \overline{\bigcup_{j=n}^\infty V_j}.
\]

Then \( W_n \) is open and closed. By Lemma 3, the set \( \bigcap_{n=1}^\infty W_n \) is not contained in \( \tilde{N}(f) \). Take a point \( \zeta_0 \) in \( \bigcap_{n=1}^\infty W_n \setminus \tilde{N}(f) \), and take an open and closed subset \( U \) of \( M(L^\infty) \) with \( \zeta_0 \in U \) and \( U \cap \tilde{N}(f) = \emptyset \). Then there is a subsequence \( \{n_j\}_j \) such that

\[
V_{n_j} \cap U \neq \emptyset \quad \text{for } j = 1, 2, \ldots.
\]

Take a point \( x_{n_j} \in V_{n_j} \cap U \). Then \( \{x_{n_j}\}_j \) satisfies our assertion.

Here we show

**Proposition 1.** Korovkin theorem does not hold on \( L^\infty \) for test functions \( C \).

**Proof.** Take points \( \zeta_1 \) and \( \zeta_2 \) in \( M(L^\infty) \) such that

\[
\zeta_1 \neq \zeta_2 \text{ and } g(\zeta_i) = g(1) \quad \text{for every } g \in C.
\]

For each \( n \), let

\[
f_n(e^{i\theta}) = |e^{i\theta} + 1|^n/2^n \quad \text{for } e^{i\theta} \in \partial D \text{ and let}
\]

\[T_n f = f(\zeta_1) f_n + (1 - f_n) f \quad \text{for } f \in L^\infty.
\]

Then \( \{T_n\}_n \) is a sequence of bounded linear operators on \( L^\infty \) with \( \|T_n\| = 1 \). Since \( T_n f = f + f_n (f(\zeta_1) - f) \), it is not difficult to see that

\[
\|T_n g - g\|_\infty \to 0 \quad \text{for } g \in C.
\]

Take \( f_0 \in L^\infty \) with \( f_0(\zeta_1) = 0 \) and \( f_0(\zeta_2) = 1 \). Then

\[
\|T_n f_0 - f_0\|_\infty = \|f_n f_0\|_\infty \geq |f_n(\zeta_2) f_0(\zeta_2)| = 1 \quad \text{for every } n.
\]
3. THEOREMS

The following is the main theorem in this paper. The proof is similar to the one of Theorem 2 in [7]. But our proof is deeply concerned with the property of \( \hat{N}(f) \) for \( f \in L^\infty \).

**Theorem 1.** Suppose \( \{T_n\}_n \) is a sequence of bounded linear operators on \( L^\infty = C(M(L^\infty)) \) such that \( \|T_n\| \to 1 \) and \( \|T_ng - g\|_\infty \to 0 \) for every \( g \in QC \) as \( n \to \infty \). Then \( \|T_nf - f\|_\infty \to 0 \) for every \( f \in L^\infty \) as \( n \to \infty \).

**Proof.** Let \( \{T_n\}_n \) be a sequence of bounded linear operators such that \( \|T_n\| \to 1 \) and \( \|T_ng - g\|_\infty \to 0 \) for \( g \in QC \). To prove that \( \|T_nf - f\|_\infty \to 0 \) for every \( f \in C(M(L^\infty)) \), suppose not. Then there exists a function \( f_0 \) in \( C(M(L^\infty)) \) with \( \|f_0\|_\infty = 1 \) and \( \sigma > 0 \) such that \( \limsup_{n \to \infty} \|T_nf_0 - f_0\|_\infty > \sigma \). By considering a subsequence, we may assume that \( \|T_nf_0 - f_0\|_\infty > \sigma \) for every \( n \). So there exists a non-empty open and closed subset \( V_n \) of \( M(L^\infty) \) such that

\[
|T_nf_0 - f_0| > \sigma \quad \text{on} \quad V_n.
\]

By Lemma 3, we may assume that \( V_n \cap \hat{N}(f_0) = \emptyset \). By Lemma 4, considering a subsequence we may assume the existence of \( x_n \) in \( V_n \) such that

\[
(x_n)_n \cap \hat{N}(f_0) = \emptyset.
\]

By (1), we have

\[
|(T_nf_0)(x_n) - f_0(x_n)| > \sigma.
\]

Also considering a subsequence, moreover we may assume that

\[
(T_nf_0)(x_n) \to a \quad \text{and} \quad f_0(x_n) \to b \quad \text{as} \quad n \to \infty.
\]

Since \( \|T_n\| \to 1 \) and \( \|f_0\|_\infty = 1 \), \( |a| \leq 1 \) and \( |b| \leq 1 \). Also by (3), \( |a - b| \geq \sigma \). Here we can find a complex number \( c \) such that

\[
|b - c| \leq 1 \quad \text{and} \quad |a - c| \geq 1 + \sigma.
\]

Since \( |a| \leq 1 \), we have \( c \neq 0 \).

Let \( w_0 \) be one of cluster points of \( \{x_n\}_n \) in \( M(L^\infty) \). By (2), \( w_0 \notin \hat{N}(f_0) \). Then by Lemma 2, there exists an open subset \( W \) of \( M(L^\infty) \) such that

\[
w_0 \in W, \quad W = \cup \{Q_g; y \in W\} \quad \text{and} \quad \overline{W} \cap \hat{N}(f_0) = \emptyset.
\]

By (4), \( f_0(w_0) = b \). By (6), there exists a function \( h \) in \( QC \) such that

\[
h(w_0) = 1 \quad \text{and} \quad h = 0 \quad \text{on} \quad \hat{N}(f_0).
\]

Then by Lemma 1 and the definition of \( \hat{N}(f_0) \), we have \( hf_0 \in QC \). Hence by taking a smaller subset of \( W \), we may assume that

\[
\|f_0 - b\|_W < \sigma/2.
\]

By (6) and \( c \neq 0 \), there exists a function \( F \) in \( QC \) such that

\[
0 \leq F/c \leq 1 \quad \text{on} \quad M(L^\infty), \quad F(w_0) = c \quad \text{and} \quad F = 0 \quad \text{on} \quad W^c.
\]

Since \( \|f_0\|_\infty = 1 \), by (5) and (8) it is not difficult to see that

\[
\|F - f_0\|_\infty < \sigma/2 + 1.
\]

Since \( \|T_n\| \to 1 \), we have

\[
\limsup_{n \to \infty} \|T_nF - T_nf_0\|_\infty < \sigma/2 + 1.
\]
Since $F \in QC$, by our assumption we have $\|T_n F - F\|_\infty \to 0$. Hence we obtain
\begin{equation}
\lim_{n \to \infty} \sup \|F - T_n f_0\|_\infty < \sigma / 2 + 1.
\end{equation}

Since $w_0$ is a cluster point of $\{x_n\}_n$, there exists a subsequence $\{x_{n_j}\}_j$ of $\{x_n\}_n$ such that
\begin{equation}
F(x_{n_j}) \to F(w_0) \text{ as } j \to \infty.
\end{equation}

Therefore we have
\begin{align*}
\lim_{n \to \infty} \sup \|F - T_n f_0\|_\infty &\geq \limsup_{j \to \infty} |F(x_{n_j}) - (T_{n_j} f_0)(x_{n_j})| \\
&= |c - a| \text{ by (4), (9) and (12)} \\
&\geq 1 + \sigma \text{ by (5)}.
\end{align*}

This contradicts (11). Thus we get our assertion.

The key point of the proof of Theorem 1 is the following: for $f \in L^\infty$ the union set of all $QC$-level sets on which $f$ have non-zero oscillations is a very small subset of $M(L^\infty)$. If its union set occupies a very big part of $M(L^\infty)$, we may not expect that the sequential type Korovkin theorem holds. We give an example such that the sequential type Korovkin theorem does not hold on some closed subsets of $M(L^\infty)$.

**Example.** Let $\{z_n\}_n$ be a sparse sequence in $D$, that is,
\[\lim_{k \to \infty} \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_n z} \right| = 1.\]

Let $b$ be the associated sparse Blaschke product;
\[b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n| \left| 1 - \bar{z}_n z \right|} \quad z \in D.\]

Let $Z(b) = \{x \in M(H^\infty) \setminus D; b(x) = 0\}$. Then by [4] we have that
(a) if $x, y \in Z(b)$ with $x \neq y$ then $Q_x \cap Q_y = \emptyset$,
(b) $N(b) = \bigcup\{Q_x; x \in Z(b)\}$.

We shall prove that the sequential type Korovkin theorem does not hold on $C(N(b))$ for test functions $QC|N(b)$. We note that $b$ has non-zero oscillation on $Q_x$ for each $x \in Z(b)$.

Let $\zeta \in N(b)$. By (a) and (b), there corresponds a point $\tau(\zeta)$ in $Z(b)$ such that $\zeta \in Q_{\tau(\zeta)}$. We note that $\tau : N(b) \to Z(b)$ is a continuous map. For $f \in C(N(b))$, let
\[(Tf)(\zeta) = \int_{N(b)} f d\mu_{\tau(\zeta)} \quad \text{for } \zeta \in N(b).\]

Then $Tf \in C(N(b))$ and $T$ is a bounded linear operator on $C(N(b))$ with $\|T\| = 1$. By the definition of $T$, we know that $T$ is the identity operator on $QC|N(b)$. Set $T_n = T$ for every $n$; then
\[\|T_n g - g\|_{N(b)} \to 0 \quad \text{for every } g \in QC|N(b).\]

Let $f_0 = b|N(b)$. Then $f_0 \in C(N(b))$ and
\[(T_n f_0)(\zeta) = \int_{N(b)} b d\mu_{\tau(\zeta)} = b(\tau(\zeta)) = 0 \quad \text{for } \zeta \in N(b).\]
Since \(|b| = 1\) on \(M(L^\infty)\), we have
\[
\|T_n f_0 - f_0\|_{\mathcal{N}(\tilde{b})} = 1.
\]
Hence \(\|T_n f_0 - f_0\|_{\mathcal{N}(\tilde{b})}\) does not converge to 0 as \(n \to \infty\).

In the same way as the proof of Theorem 1, we have the following theorems.

**Theorem 2.** Let \(\{T_n\}_n\) be a sequence of bounded linear operators on \(H^\infty\) such that \(\|T_n\| \to 1\) and \(\|T_n g - g\|_\infty \to 0\) for every \(g \in QA\). Then \(\|T_n f - f\|_\infty \to 0\) for every \(f \in H^\infty\).

**Proof.** We give a remark for the proof of this theorem.

(1) We cannot find \(F\) in \(QA\) which satisfies (9).

To overcome this difficulty, it is sufficient to show that for every \(\epsilon > 0\) there exists \(F\) in \(QA\) such that \(F(w_0) = c, |F| < \epsilon\) on \(W^c\) and \(|F| + |c - F| < 1 + \epsilon\) on \(M(L^\infty)\).

The existence of such an \(F\) is proved in the proof of Theorem 2 in [6] for general function algebras.

Let \(I\) be the identity operator on \(L^\infty\). For a bounded linear operator \(T\) on \(L^\infty\), let \(\|T\|_{QC} = \sup\{\|T f\|_\infty; f \in QC, \|f\|_\infty \leq 1\}\).

**Theorem 3.** Suppose \(\{T_n\}_n\) is a sequence of bounded linear operators on \(L^\infty = C(M(L^\infty))\) such that \(\|T_n\| \to 1\) and \(\|T_n - I\|_{QC} \to 0\) as \(n \to \infty\). Then \(\|T_n - I\| \to 0\) as \(n \to \infty\).

**Proof.** We give the outline. Suppose that there exists a sequence \(\{f_n\}_n\) in \(C(M(L^\infty))\) such that
\[
\|f_n\|_\infty = 1 \quad \text{and} \quad \|T_n f_n - f_n\|_\infty > \sigma > 0.
\]

By considering a subsequence, we may assume the existence of \(\{x_n\}_n\) in \(M(L^\infty)\) such that \(x_n \notin \mathcal{N}(f_n)\), \(|(T_n f_n)(x_n) - f_n(x_n)| > \sigma\), \((T_n f_n)(x_n) \to a\), and \(f_n(x_n) \to b\).

Take \(c\) with \(|b - c| \leq 1\) and \(|a - c| \geq 1 + \sigma\). Find \(F_n\) in \(QC\) such that
\[
\|F_n - f_n\|_\infty < \sigma/2 + 1 \quad \text{and} \quad F_n(x_n) = c.
\]

By our assumption, we have
\[
\limsup_{n \to \infty} \|F_n - T_n f_n\|_\infty < \sigma/2 + 1,
\]
but
\[
\|F_n - T_n f_n\|_\infty \geq |F_n(x_n) - (T_n f_n)(x_n)| \to |c - a| \geq 1 + \sigma.
\]
This is a contradiction.

**References**


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