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CROSSED PRODUCTS OF HILBERT C^*-MODULES

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Abstract. We define the notion of crossed products of Hilbert C^*-modules by regular multiplicative unitaries, which is a generalization of the notion of crossed products of C^*-algebras. Crossed products of Hilbert C^*-modules arise naturally as imprimitivity bimodules for the crossed products of Morita equivalent C^*-algebras.

Introduction

Hilbert C^*-modules and adjointable operators between them now play a very important role in the modern theory of C^*-algebras; for example in Kasparov's KK-theory (see [BS]) and in the C^*-algebraic approach to the locally compact quantum group theory (see [BS2] and [W]). Baaj and Skandalis [BS] introduced the notion of coactions of Hopf C^*-algebras on Hilbert C^*-modules, and defined the crossed product $X \times \hat{G}$ of a Hilbert C^*-module X over a C^*-algebra B by a coaction δ_B of a locally compact group G on B to be the tensor product $X \otimes_{\delta_B} (B \times \hat{G})$; see [BS, Définition 6.7]. They proved that $\mathcal{K}(X \times \hat{G})$ and $\mathcal{K}(X) \times \hat{G}$ are isomorphic; see [BS, Proposition 6.9]. We [B] modified this result and showed that if X is an A, B-imprimitivity bimodule satisfying suitable conditions, then $X \times \hat{G}$ yields an imprimitivity bimodule for the crossed products $A \times \hat{G}$ and $B \times \hat{G}$. As pointed out in [ER], this formula of crossed products has some disadvantages. For example, the description of the actions of $A \times \hat{G}$ and $B \times \hat{G}$ is not clear, and it is not obvious that $(A \times \hat{G}) \otimes_{\delta_A} X$ and $X \otimes_{\delta_B} (B \times \hat{G})$ are isomorphic. Echterhoff and Raeburn [ER] gave an answer to this problem by introducing the notion of multipliers of imprimitivity bimodules. However their solution is only a spatial one. On the other hand, Baaj and Skandalis [BS2] introduced the notion of regular multiplicative unitaries and defined the notion of crossed products of C^*-algebras by coactions of (the Hopf C^*-algebras associated to) regular multiplicative unitaries. This is a beautiful generalization of the classical notion of crossed products by coactions of locally compact groups.

In this paper we present an abstract solution to the problem about the formula for imprimitivity bimodules of the crossed products of Morita equivalent C^*-algebras. We define the crossed product of a Hilbert C^*-module by a regular multiplicative unitary as a subspace of adjointable operators between Hilbert C^*-modules. Our approach is close to the spirit of [BS]. We neither use the new machinary of [ER]

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nor need to represent all X, A and B on Hilbert spaces as in [B] and [ER]. We treat crossed products of C^*-algebras by coactions of locally compact groups as a special case, and provide an imprimitivity bimodule for the crossed products of Morita equivalent C^*-algebras with an obvious imprimitivity bimodule structure. The result [BS, Proposition 6.9] then becomes more obvious. We shall deal with regular multiplicative unitaries instead of locally compact groups. With a suitable choice of multiplicative unitaries, we obtain the results [BS, Proposition 6.9], [B, Theorem 2.16] and [ER, Theorem 3.2].

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§1. Crossed products of Hilbert C^*-modules

Throughout this paper, H is a Hilbert space, B is a C^*-algebra, and X is a right Hilbert C^*-module over B. Tensor products of C^*-algebras will be assumed minimal. The reader is refered to [B, §2] for the definition of nondegenerate linear maps and compatible linear maps between Hilbert C^*-modules, and to [B2, §2] for tensor products of such maps.

We recall some notation of Hilbert C^*-modules; for more details see [Bl, Chapter VI, §13] and [L]. If Y and Z are right Hilbert C^*-modules (over the same C^*-algebra), we denote by $\mathcal{L}(Y,Z)$ the space of adjointable operators from Y to Z, and denote by $\mathcal{K}(Y,Z)$ the subspace of compact operators.

Let Y be a right Hilbert C-module and let Z be a right Hilbert D-module. The algebraic tensor product $Y \odot Z$ is a right pre-Hilbert C^*-module over $C \odot D$ in the obvious way. Let $Y \hat{\odot} Z$ denote the quotient of $Y \odot Z$ by the subspace of vectors of length zero. Then the completion $Y \hat{\otimes} Z$ of $Y \hat{\odot} Z$ is a a right Hilbert C^*-module over the minimal tensor product $C \otimes D$. The image of each element $\sum_i y_i \otimes z_i \in Y \odot Z$ under the canonical quotient map is denoted by $\sum_i y_i \hat{\otimes} z_i$. Let $\mathcal{V}$ be another right Hilbert C-module and let $\mathcal{W}$ be another right Hilbert D-module. Then for any $S \in \mathcal{L}(Y,\mathcal{V})$ and $T \in \mathcal{L}(Z,\mathcal{W})$, there is an adjointable operator $S \hat{\otimes} T$ from $Y \hat{\otimes} Z$ into $\mathcal{V} \hat{\otimes} \mathcal{W}$ such that $(S \hat{\otimes} T)(y \hat{\otimes} z) = S(y) \hat{\otimes} T(z)$ for all $y \in Y$, $z \in Z$.

Let Γ be a regular multiplicative unitary of H in the sense of [BS2, Définition 3.3]. We denote by $\mathcal{L}(H)_*$ the predual of the algebra $\mathcal{L}(H)$. Put

$$\ell(\omega) \;=\; (\omega \otimes id)(\Gamma), \quad \mu(\omega) \;=\; (id \otimes \omega)(\Gamma), \qquad \forall \omega \in \mathcal{L}(H)_*.$$

Then $A(\Gamma) = \{\ell(\omega) : \omega \in \mathcal{L}(H)_*\}$ is an algebra, and the closure $\mathcal{S}$ of $A(\Gamma)$ is a nondegenerate C^*-subalgebra of $\mathcal{L}(H)$. The closure of $\widehat{\mathcal{S}}$ of $\hat{A}(\Gamma) = \{\mu(\omega) : \omega \in \mathcal{L}(H)_*\}$ is also a nondegenerate C^*-subalgebra of $\mathcal{L}(H)$. Put

$$\delta_\Gamma(f) \;=\; \Gamma(f \otimes 1)\Gamma^*, \qquad \forall f \in \mathcal{S}.$$

Then $(\mathcal{S}, \delta_\Gamma)$ is a bisimplifiable Hopf C^*-algebra; see [BS2, Théorème 3.8].

For any $\phi \in \mathcal{L}(H)_*$ and $f \in \mathcal{S}$, we define

$$(\phi \cdot f)(T) \;=\; \phi(fT), \quad (f \cdot \phi)(T) \;=\; \phi(Tf), \qquad \forall T \in \mathcal{L}(H).$$

Recall from [S, Corollary 1.15.5] that the linear span of $\{\, \omega_{u,v} \,:\, u,v \in H\}$ is dense in $\mathcal{L}(H)_*$. Here $\omega_{u,v}$ is defined by $\omega_{u,v}(T) = \langle Tv|u\rangle$ for all $T \in \mathcal{L}(H)$. Since $\mathcal{S}$ is a nondegenerate subalgebra, by [DW, Theorem 16.1] each $u \in H$ can be written as $u = gt$ for some $g \in \mathcal{S}$ and $t \in H$. Thus $\omega_{u,v} = \omega_{t,v} \cdot g^*$, and hence the linear span of $\{\, \phi \cdot f \,:\, f \in \mathcal{S},\ \phi \in \mathcal{L}(H)_* \,\}$ is dense in $\mathcal{L}(H)_*$. By [DW, Theorem 16.1] each

$\omega \in \mathcal{L}(H)_*$ can be written as $\omega = \phi \cdot f$ for some $f \in \mathcal{S}$ and $\phi \in \mathcal{L}(H)_*$. By a similar argument, each $\omega \in \mathcal{L}(H)_*$ can be written as $\omega = f \cdot \phi$.

Let $id_B : B \longrightarrow \mathcal{L}(B)$ and $id_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathcal{L}(\mathcal{K}(H))$ denote the homomorphisms defined by $id_B(b)c = bc$ and $id_{\mathcal{S}}(f)\theta = f\theta$ for all $b, c \in B$, $f \in \mathcal{S}$ and $\theta \in \mathcal{K}(H)$. The homomorphism $id_B \otimes id_{\mathcal{S}}$ from $B \otimes \mathcal{S}$ into $\mathcal{L}(B \otimes \mathcal{K}(H))$, given by $(id_B \otimes id_{\mathcal{S}})(\beta)\alpha = \beta\alpha$ for all $\alpha \in B \otimes \mathcal{K}(H)$ and $\beta \in B \otimes \mathcal{S}$, is nondegenerate and injective. To prove that $id_B \otimes id_{\mathcal{S}}$ is injective, we represent B faithfully and nondegenerately on a Hilbert space H_1, and hence we can view $B \otimes \mathcal{K}(H)$ and $B \otimes \mathcal{S}$ as nondegenerate C^*-subalgebras of $\mathcal{L}(H_1 \otimes H)$. If $(id_B \otimes id_{\mathcal{S}})(\beta) = 0$, then $\beta\alpha = 0$ for all $\alpha \in B \otimes \mathcal{K}(H)$. For each $u \in H_1 \otimes H$, we write $u = \alpha v$ for some $\alpha \in B \otimes \mathcal{K}(H)$ and $v \in H_1 \otimes H$. It then follows that $\beta u = (\beta\alpha)v = 0$, and hence $\beta = 0$. Now by [B, Proposition 2.3] $id_B \otimes id_{\mathcal{S}}$ extends to a homomorphism Υ from $\mathcal{L}(B \otimes \mathcal{S})$ into $\mathcal{L}(B \otimes \mathcal{K}(H))$. Let $\pi_X : X \longrightarrow \mathcal{L}(B, X)$ be the linear map defined by $\pi_X(x)b = xb$ for all $x \in X$ and $b \in B$. The linear map $\pi_X \hat{\otimes} id_{\mathcal{S}}$ from $X \hat{\otimes} \mathcal{S}$ into $\mathcal{L}(B \otimes \mathcal{K}(H), X \hat{\otimes} \mathcal{K}(H))$, given by $(\pi_X \hat{\otimes} id_{\mathcal{S}})(\xi)\alpha = \xi\alpha$ for all $\alpha \in B \otimes \mathcal{K}(H)$ and $\xi \in X \hat{\otimes} \mathcal{S}$, is nondegenerate and compatible with $id_B \otimes id_{\mathcal{S}}$; see [B, Definition 2.1]. The existence of the tensor product maps $id_B \otimes id_{\mathcal{S}}$ and $\pi_X \hat{\otimes} id_{\mathcal{S}}$ can also be found in [B2, Lemma 2.4]. By [B, Proposition 2.3] $\pi_X \hat{\otimes} id_{\mathcal{S}}$ extends to a linear map Π from $\mathcal{L}(B \otimes \mathcal{S}, X \hat{\otimes} \mathcal{S})$ into $\mathcal{L}(B \otimes \mathcal{K}(H), X \hat{\otimes} \mathcal{K}(H))$. Since Π is compatible with Υ and Υ is injective, Π is injective.

We summarize the above discussion in the following result.

Proposition 1.1. *There exist an injective homomorphism Υ from $\mathcal{L}(B \otimes \mathcal{S})$ into $\mathcal{L}(B \otimes \mathcal{K}(H))$, and an injective Υ-compatible linear map Π from $\mathcal{L}(B \otimes \mathcal{S}, X \hat{\otimes} \mathcal{S})$ into $\mathcal{L}(B \otimes \mathcal{K}(H), X \hat{\otimes} \mathcal{K}(H))$. They are defined by*

$$\Upsilon(R)(\beta\alpha) \;=\; (R\beta)\alpha, \qquad \Pi(T)(\beta\alpha) \;=\; (T\beta)\alpha,$$

for all $R \in \mathcal{L}(B \otimes \mathcal{S})$, $T \in \mathcal{L}(B \otimes \mathcal{S}, X \hat{\otimes} \mathcal{S})$, $\beta \in B \otimes \mathcal{S}$ and $\alpha \in B \otimes \mathcal{K}(H)$.

Therefore we view $\mathcal{L}(B \otimes \mathcal{S}, X \hat{\otimes} \mathcal{S})$ as a subspace of $\mathcal{L}(B \otimes \mathcal{K}(H), X \hat{\otimes} \mathcal{K}(H))$, and we view $\mathcal{L}(B \otimes \mathcal{S})$ as a subalgebra of $\mathcal{L}(B \otimes \mathcal{K}(H))$. We will also view $\mathcal{L}(B \otimes \widehat{\mathcal{S}})$ as a subalgebra of $\mathcal{L}(B \otimes \mathcal{K}(H))$, and we view $\mathcal{L}(\mathcal{K}(X) \otimes \widehat{\mathcal{S}})$ as a subalgebra of $\mathcal{L}(\mathcal{K}(X) \otimes \mathcal{K}(H)) = \mathcal{L}(X \hat{\otimes} \mathcal{K}(H))$.

We define maps P_1 from $X \oplus B$ into X, and P_2 from $X \oplus B$ into B by

$$P_1(x \oplus b) \;=\; x, \qquad P_2(x \oplus b) \;=\; b, \qquad \forall x \in X,\ \forall b \in B.$$

Then we define maps $\bar{c}_{ij}$ from $P_i \mathcal{L}(X \oplus B) P_j^*$ into $\mathcal{L}(X \oplus B)$ by

$$\bar{c}_{ij}(T_{ij}) \;=\; P_i^* T_{ij} P_j, \qquad \forall T_{ij} \in P_i \mathcal{L}(X \oplus B) P_j^*.$$

In a similar fashion, we define maps Q_i and $\bar{d}_{ij}$ with $X \hat{\otimes} \mathcal{K}(H)$ and $B \otimes \mathcal{K}(H)$ in place of X and B. Note that

$$(X \oplus B) \hat{\otimes} \mathcal{K}(H) \;=\; [X \hat{\otimes} \mathcal{K}(H)] \oplus [B \otimes \mathcal{K}(H)],$$

and $Q_i = P_i \hat{\otimes} 1$. Thus the restriction of $\bar{d}_{11}$ to $\mathcal{K}(X \hat{\otimes} \mathcal{S})$ is $c_{11} \otimes id_{\mathcal{S}}$, the restriction of $\bar{d}_{22}$ to $B \otimes \mathcal{S}$ is $c_{22} \otimes id_{\mathcal{S}}$, and the restriction of $\bar{d}_{12}$ to $X \hat{\otimes} \mathcal{S}$ is $c_{12} \hat{\otimes} id_{\mathcal{S}}$.

We recall from [BS, Définition 2.1] that $\tilde{M}(X \hat{\otimes} \mathcal{S})$ is the space of $T \in \mathcal{L}(B \otimes \mathcal{S}, X \hat{\otimes} \mathcal{S})$ such that $(1_X \hat{\otimes} f)T$, $T(1_B \otimes f) \in X \hat{\otimes} \mathcal{S}$ for all $f \in \mathcal{S}$. Let $\delta_B : B \longrightarrow \tilde{M}(B \otimes \mathcal{S})$ be a coaction of $\mathcal{S}$ on B. Following [BS, Définition 2.2], a δ_B-compatible coaction of $\mathcal{S}$ on X is defined to be a nondegenerate linear map $\delta_X : X \longrightarrow \tilde{M}(X \hat{\otimes} \mathcal{S})$ which is compatible with δ_B and satisfies the condition

$$(\delta_X \hat{\otimes} id_{\mathcal{S}})^{-} \circ \delta_X \;=\; (id_X \hat{\otimes} \delta_\Gamma)^{-} \circ \delta_X,$$

as maps from X into $\mathcal{L}(B \otimes \mathcal{S} \otimes \mathcal{S}, X \hat{\otimes} \mathcal{S} \hat{\otimes} \mathcal{S})$. Put $J = \mathcal{K}(X \oplus B)$ and $E = \mathcal{K}(X)$. Then there exist a coaction $\delta_E : E \longrightarrow \tilde{M}(E \otimes \mathcal{S})$ of $\mathcal{S}$ on E and a coaction $\delta_J : J \longrightarrow \tilde{M}(J \otimes \mathcal{S})$ of $\mathcal{S}$ on J such that

$$\begin{aligned} \delta_X(\theta x) &= \delta_E(\theta)\delta_X(x), \qquad \forall \theta \in E,\ \forall x \in X; \\ \delta_J \circ c_{22} &= (c_{22} \otimes id)^{-} \circ \delta_B, \\ \delta_J \circ c_{12} &= (c_{12} \hat{\otimes} id)^{-} \circ \delta_X. \end{aligned}$$

The reader is refered to [BS, §2] and [B, §2] for the existence of δ_E and δ_J.

Definition 1.2. Suppose that $\delta_B : B \longrightarrow \tilde{M}(B \otimes \mathcal{S})$ is a coaction of $\mathcal{S}$ on B, and $\delta_X : X \longrightarrow \tilde{M}(X \hat{\otimes} \mathcal{S})$ is a δ_B-compatible coaction of $\mathcal{S}$ on X. The crossed product $X \times_{\delta_X} \Gamma$ of X by Γ is the closed subspace of $\mathcal{L}(B \otimes \mathcal{K}(H), X \hat{\otimes} \mathcal{K}(H))$ generated by

$$\delta_X(x)[1_B \otimes f], \qquad \forall x \in X,\ \forall f \in \widehat{\mathcal{S}}.$$

Proposition 1.3. *(i) Each element* $[1_E \otimes \mu(\omega)]\delta_X(x)$ *is the limit of finite sums*

$$\sum_{i=1}^{n} \delta_X(y_i)[1_B \otimes \mu(\phi_i)], \qquad y_i \in X,\ \phi_i \in \mathcal{L}(H)_*.$$

(ii) Each element $\delta_X(x)[1_B \otimes \mu(\omega)^*]$ *is the limit of finite sums*

$$\sum_{i=1}^{n} [1_E \otimes \mu(\phi_i)^*]\delta_X(y_i), \qquad y_i \in X,\ \phi_i \in \mathcal{L}(H)_*.$$

Proof. (i) We write $\omega = \phi \cdot g$ for some $\phi \in \mathcal{L}(H)_*$ and $g \in \mathcal{S}$. We shall denote $\bar{c}_{ij}(m)$ by m^c. Since $[1_E \otimes g]\delta_X(x) \in X \hat{\otimes} \mathcal{S}$, it is the limit of finite sums $\sum_i y_i \hat{\otimes} h_i$. We then compute

$$\begin{aligned} \bar{d}_{12}\{[1_E \otimes \mu(\omega)]\delta_X(x)\} &= [1_E^c \otimes \mu(\omega)]\delta_J(x^c) \\ &= [1_J \otimes \mu(\omega)]\delta_J(x^c) \\ &= (id \otimes id \otimes \omega)([1_J \otimes \Gamma][\delta_J(x^c) \otimes 1]) \\ &= (id \otimes id \otimes \omega)((id \otimes \delta_\Gamma)^{-} \circ \delta_J(x^c)[1_J \otimes \Gamma]) \\ &= (id \otimes id \otimes \phi)([1_J \otimes 1_{\mathcal{S}} \otimes g](\delta_J \otimes id)^{-} \circ \delta_J(x^c)[1_J \otimes \Gamma]) \\ &= (id \otimes id \otimes \phi)((\delta_J \otimes id)([1_J \otimes g]\delta_J(x^c))[1_J \otimes \Gamma]) \\ &= (id \otimes id \otimes \phi)((\delta_J \otimes id) \circ (c_{12} \hat{\otimes} id)([1_E \otimes g]\delta_X(x))[1_J \otimes \Gamma]), \end{aligned}$$

which is the limit of finite sums

$$\sum_{i=1}^{n}(id\otimes id\otimes\phi)((\delta_J\otimes id)\circ(c_{12}\hat{\otimes}id)(y_i\hat{\otimes}h_i)[1_J\otimes\Gamma])$$
$$= \sum_{i=1}^{n}(id\otimes id\otimes\phi\cdot h_i)([\delta_J(y_i^c)\otimes 1][1_J\otimes\Gamma])$$
$$= \sum_{i=1}^{n}\delta_J(y_i^c)[1_J\otimes\mu(\phi\cdot h_i)]$$
$$= \bar{d}_{12}\{\sum_{i=1}^{n}\delta_X(y_i)[1_B\otimes\mu(\phi\cdot h_i)]\}.$$

Since $\bar{d}_{12}$ is an isometry we get the desired result.

(ii) We write $\omega^* = g\cdot\phi$ for some $\phi\in\mathcal{L}(H)_*$ and $g\in\mathcal{S}$. Since $\delta_X(x)[1_B\otimes g]\in X\hat{\otimes}\mathcal{S}$, it is the limit of finite sums $\sum_i y_i\hat{\otimes}h_i$. We then compute

$$\begin{aligned}\bar{d}_{12}\{\delta_X(x)[1_B\otimes\mu(\omega)^*]\} &= \delta_J(x^c)[1_B^c\otimes\mu(\omega)^*]\\ &= \delta_J(x^c)[1_J\otimes\mu(\omega)^*]\\ &= (id\otimes id\otimes\omega^*)([\delta_J(x^c)\otimes 1][1_J\otimes\Gamma^*])\\ &= (id\otimes id\otimes\omega^*)([1_J\otimes\Gamma^*](id\otimes\delta_\Gamma)\bar{)}\circ\delta_J(x^c))\\ &= (id\otimes id\otimes\phi)([1_J\otimes\Gamma^*](\delta_J\otimes id\bar{)}\circ\delta_J(x^c)[1_J\otimes 1_{\mathcal{S}}\otimes g])\\ &= (id\otimes id\otimes\phi)([1_J\otimes\Gamma^*](\delta_J\otimes id)(\delta_J(x^c)[1_J\otimes g]))\\ &= (id\otimes id\otimes\phi)([1_J\otimes\Gamma^*](\delta_J\otimes id)\circ(c_{12}\hat{\otimes}id)(\delta_X(x)[1_B\otimes g])),\end{aligned}$$

which is the limit of finite sums

$$\sum_{i=1}^{n}(id\otimes id\otimes\phi)([1_J\otimes\Gamma^*](\delta_J\otimes id)\circ(c_{12}\hat{\otimes}id)(y_i\hat{\otimes}h_i))$$
$$= \sum_{i=1}^{n}(id\otimes id\otimes h_i\cdot\phi)([1_J\otimes\Gamma^*][\delta_J(y_i^c)\otimes 1])$$
$$= \sum_{i=1}^{n}[1_J\otimes\mu(\phi_i)^*]\delta_J(y_i^c)$$
$$= \bar{d}_{12}\{\sum_{i=1}^{n}[1_E\otimes\mu(\phi_i)^*]\delta_X(y_i)\},$$

where $\phi_i = (h_i\cdot\phi)^*$. □

Proposition 1.4. *The crossed product $X\times_{\delta_X}\Gamma$ is the closed subspace of $\mathcal{L}(B\otimes\mathcal{K}(H), X\hat{\otimes}\mathcal{K}(H))$ generated by*

(i) $[1_E\otimes f]\delta_X(x)$, $\quad\forall x\in X,\ \forall f\in\widehat{\mathcal{S}}$*;*

(ii) $[1_E\otimes f]\delta_X(x)[1_B\otimes g]$, $\quad\forall x\in X,\ f,g\in\widehat{\mathcal{S}}$*.*

Proof. (i) This is a consequence of Proposition 1.3.

(ii) Since $\widehat{\mathcal{S}}$ is a C^*-algebra, each $h \in \widehat{\mathcal{S}}$ is the limit of products $\mu(\phi)^*\mu(\psi)$. It follows from Proposition 1.3(ii) that each $\delta_X(x)[1_B \otimes h]$ is the limit of finite sums

$$\sum_{i=1}^{n}[1_E \otimes \mu(\phi_i)^*]\delta_X(y_i)[1_B \otimes \mu(\psi_i)].$$

□

Suppose that D is a C^*-algebra and $\delta_D : D \longrightarrow \tilde{M}(D \otimes \mathcal{S})$ is a coaction of $\mathcal{S}$ on D. If we view D as a Hilbert D-module, then δ_D is a δ_D-compatible coaction of $\mathcal{S}$ on D.

Proposition 1.5. *The crossed product $D \times_{\delta_D} \Gamma$ is a C^*-subalgebra of $\mathcal{L}(D \otimes \mathcal{K}(H)) = \mathcal{L}(D \hat{\otimes} H)$.*

Proof. For any $x, y \in D$ and $f, g \in \widehat{\mathcal{S}}$, we have

$$\begin{aligned} [1_D \otimes f]\delta_D(x)\delta_D(y)[1_E \otimes g] &= [1_D \otimes f]\delta_D(xy)[1_D \otimes g], \\ \{\delta_D(x)[1_D \otimes f]\}^* &= [1_D \otimes f^*]\delta_D(x^*). \end{aligned}$$

Use Proposition 1.4 we get the desired result. □

Note that the above proposition treats [LPRS, Definition 2.4] as a special case since $\mathcal{L}(D \otimes \mathcal{K}(H)) = M(D \otimes \mathcal{K}(H))$.

Recall from [Bl, Definition 13.1.1] that a Hilbert C^*-module is full if the inner product on it is full. Put

$$\mathcal{E} = E \times_{\delta_E} \Gamma, \qquad \mathcal{B} = B \times_{\delta_B} \Gamma, \qquad \mathcal{X} = X \times_{\delta_X} \Gamma.$$

Theorem 1.6. *Suppose that X is a full right Hibert B-module. Then $\mathcal{X}$ is an $\mathcal{E}, \mathcal{B}$-imprimitivity bimodule. The left and right actions of $\mathcal{E}$ and $\mathcal{B}$ on $\mathcal{X}$ are the composition of operators between Hilbert C^*-modules, and the inner products are given by*

$${}_{\mathcal{E}}\langle S|T\rangle = ST^*, \quad \langle T|S\rangle_{\mathcal{B}} = T^*S, \qquad \forall S, T \in \mathcal{X}.$$

Proof. For any $x, y \in X$ and $f, g \in \widehat{\mathcal{S}}$, we have

$$\begin{aligned} [1_E \otimes f]\delta_X(x)\{[1_E \otimes g]\delta_X(y)\}^* &= [1_E \otimes f]\delta_E(\theta_{x,y})[1_E \otimes g^*], \\ \{\delta_X(y)[1_B \otimes g]\}^*\delta_X(x)[1_B \otimes f] &= [1_B \otimes g^*]\delta_B(\langle x|y\rangle_B)[1_B \otimes f]. \end{aligned}$$

By Proposition 1.4(ii) and Proposition 1.5, ${}_{\mathcal{E}}\langle\cdot|\cdot\rangle$ and $\langle\cdot|\cdot\rangle_{\mathcal{B}}$ are full inner products on $\mathcal{X}$. The other assertions follow from routine computations. □

§2. Morita equivalence of crossed products

In this section X is an A, B-imprimitivity bimodule, and δ_A and δ_B are coactions of $\mathcal{S}$ on A and B. We keep the notation of Section 1.

We recall from [B, Definition 2.15] the following definition.

Definition 2.1. A δ_A, δ_B-compatible coaction of $\mathcal{S}$ on X is a δ_B-compatible coaction δ_X of $\mathcal{S}$ on X satisfying the condition

$$\delta_X(x)\delta_X(y)^* = (\vartheta \otimes \bar{id}) \circ \delta_A({}_A\langle x|y\rangle), \qquad \forall x, y \in X, \tag{1}$$

where $\vartheta : A \longrightarrow \mathcal{K}(X)$ is the natural isomorphism. The coactions δ_A and δ_B are said to be (strongly) Morita equivalent by means of the imprimitivity system (X, δ_X).

Put $\mathcal{A} = A \times_{\delta_A} \Gamma$, $\mathcal{B} = B \times_{\delta_B} \Gamma$, $\mathcal{E} = E \times_{\delta_E} \Gamma$, and $\mathcal{X} = X \times_{\delta_X} \Gamma$.

Both [B, Theorem 2.16] and [ER, Theorem 3.2] were obtained by representing all X, A and B on Hilbert spaces. The following result is an abstract version of them.

Theorem 2.2. *Suppose that δ_A and δ_B are Morita equivalent by means of an imprimitivity system (X, δ_X). Then the map $(\vartheta \otimes \bar{id})$ defines an isomorphism of C^*-algebras $\mathcal{A}$ and $\mathcal{E}$. The crossed product $\mathcal{X}$ is an $\mathcal{A},\mathcal{B}$-imprimitivity bimodule. The imprimitivity bimodule structure on $\mathcal{X}$ is given by*

$$\begin{aligned} \alpha \cdot T &= (\vartheta \otimes \bar{id})(\alpha)T, \quad T \cdot \beta \;=\; T\beta, \\ (\vartheta \otimes \bar{id})({}_{\mathcal{A}}\langle S|T\rangle) &= ST^*, \quad \langle T|S\rangle_{\mathcal{B}} \;=\; T^*S, \end{aligned}$$

for all $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $S, T \in \mathcal{X}$.

Proof. Since

$$\delta_E(\theta_{x,y}) \;=\; \delta_X(x)\delta_X(y)^* \;=\; (\vartheta \otimes \bar{id}) \circ \delta_A({}_A\langle x|y\rangle),$$

for all $x, y \in X$, it follows that

$$\delta_E \circ \vartheta \;=\; (\vartheta \otimes \bar{id}) \circ \delta_A.$$

Thus the map $(\vartheta \otimes \bar{id})$ defines an isomorphism of C^*-algebras $\mathcal{A}$ and $\mathcal{E}$. The other assertions follow from Theorem 1.6. □

Remark 2.3. (a) We put

$$\begin{aligned} \Phi(\alpha \otimes_{\delta_A} x) &= (\vartheta \otimes \bar{id})(\alpha)\delta_X(x), \qquad \forall x \in X,\ \forall \alpha \in \mathcal{A}, \\ \Psi(x \otimes_{\delta_B} \beta) &= \delta_X(x)\beta, \qquad \forall x \in X,\ \forall \beta \in \mathcal{B}. \end{aligned}$$

Then Φ defines an isomorphism of left Hilbert $\mathcal{A}$-modules $\mathcal{A} \otimes_{\delta_A} X$ and $\mathcal{X}$, and Ψ defines an isomorphism of right Hilbert $\mathcal{B}$-modules $X \otimes_{\delta_B} \mathcal{B}$ and $\mathcal{X}$.

(b) Suppose that G is a locally compact group with a left Haar measure. Then the operator W_G on $L^2(G \times G)$ defined by $(W_G\xi)(s,t) = \xi(s, s^{-1}t)$ is a regular multiplicative unitary. The Hopf C^*-algebra $\mathcal{S}_{W_G}$ associated with W_G is the reduced group C^*-algebra $C_r^*(G)$. The crossed product $B \times_{\delta_B} W_G$ is just the crossed product $B \times_{\delta_B} G$ of [LPRS, Definition 2.4]. Therefore the results [BS, Proposition 6.9], [B, Theorem 2.16] and [ER, Theorem 3.2] are special cases of Theorem 2.2.

(c) If V_G is defined as in [BS2, Example 1.2(2)], then $\mathcal{S}_{V_G} = C_0(G)$. On the other hand, we note that $\widehat{\mathcal{S}}_{W_G} = C_0(G)$. Take G to be a compact group; then V_G is compact and W_G is discrete in the sense of [BS2, Définition 1.7]. Thus this definition seems to be vague as it does not characterize the type of groups in terms of the type of multiplicative unitaries.

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