

A UNIFORM L^p ESTIMATE OF BESSEL FUNCTIONS AND DISTRIBUTIONS SUPPORTED ON S^{n-1}

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ABSTRACT. A uniform L^p estimate of Bessel functions is obtained, which is used to get a characterization of the L^2 measures on the unit sphere of R^n in terms of the mixed L^p norm of the Fourier transform of the measures.

1. INTRODUCTION

For $n \geq 2$, let $S(R^n)$ be the space of Schwartz class functions. The dual space of $S(R^n)$ is denoted by $S'(R^n)$ and called the tempered distributions. Let S^{n-1} be the unit sphere of R^n . For $\mu \in S'(R^n)$, we list the following statements.

(A) μ is a finite measure on S^{n-1} .

(B) $\langle \mu, f \rangle = 0$ for $f(x) \in S(R^n)$, vanishing on S^{n-1} .

(C) μ is supported on S^{n-1} .

It is easy to see that (A) \rightarrow (B) \rightarrow (C). An example of Schwartz [8] indicates that in general (C) \nrightarrow (B), while the Hilbert transform distribution on a piece of S^1 could be used to show that in general (B) \nrightarrow (A). One motivation for us to study these properties comes from the area of partial differential equations with constant coefficients. For example, if $\hat{\mu}$ is a solution of the equation $\Delta u(x) + u(x) = 0$, then μ satisfies the property (B). In [3], Hörmander proved that if $\hat{\mu} \in L^p(R^n)$, $1 \leq p \leq \frac{2n}{n-1}$, then (C) implies $\mu = 0$. For $p > \frac{2n}{n-1}$, we have

Theorem 1.1 ([2], Lemma 1, page 504). *If*

$$\begin{aligned} \hat{\mu} \in L^p(R^n), \quad 4 < p \leq \infty, \quad n = 2, \\ 3 < p < \infty, \quad n = 3, \quad \frac{2n}{n-1} < p < \frac{2n}{n-3}, \quad n > 3, \end{aligned}$$

then (C) implies (B).

A natural question is: When does (C) imply (A)? First we notice that since S^{n-1} is compact, the fact that $\hat{\mu} \in L^{p_0}(R^n)$ for some $p_0 \geq 1$ implies that all derivatives of $\hat{\mu}$ are also in the space $L^{p_0}(R^n)$. Thus the Sobolev embedding lemma ([9], Theorem 0.3.7, page 26) yields that $\hat{\mu} \in L^\infty(R^n)$ and hence $\hat{\mu} \in L^p(R^n)$ for all $p > p_0$. Therefore if one seeks L^p conditions on $\hat{\mu}$, then a smaller p means a

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stronger condition. To find a precise condition on $\hat{\mu}$, let us study the measure μ on S^{n-1} given by $\mu = g(s) ds$, where $g(s)$ is a smooth function on S^{n-1} and ds is the area measure of S^{n-1} . In this special case, it is well known [7] that $\hat{\mu}(\xi)$ is essentially a constant multiple of $(1 + |\xi|)^{-\frac{n-1}{2}}$, which implies the following facts:

$$(1) \quad \liminf_{p \rightarrow (\frac{2n}{n-1})^+} \left(p - \frac{2n}{n-1} \right)^{\frac{1}{p}} \|\hat{\mu}\|_{L^p(R^n)} < \infty,$$

$$(2) \quad \limsup_{R \rightarrow \infty} \left\{ \frac{1}{R} \int_{|\xi| < R} |\hat{\mu}(\xi)|^2 d\xi \right\} < \infty.$$

Previously, Agmon and Hörmander ([1], Theorems 2.1 and 2.2) found that (2) is true if and only if μ is a L^2 density on S^{n-1} . Their proof uses the Plancherel theorem and some smooth cut-off functions.

For $x \in R^n$, denote its polar coordinates by (r, x') . Define the mixed norm spaces $L^p(R^+)(L^2(S^{n-1}))$ by

$$\begin{aligned} L^p(R^+)(L^2(S^{n-1})) &= \left\{ f(r, x'), \|f\|_{L^p(L^2)} \right. \\ &= \left. \left(\int_0^\infty \left(\int_{S^{n-1}} |f(r, x')|^2 dx' \right)^{\frac{p}{2}} r^{n-1} dr \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

We list the following statement for $L^p(L^2)$:

$$(3) \quad \liminf_{p \rightarrow (\frac{2n}{n-1})^+} \left(p - \frac{2n}{n-1} \right)^{\frac{1}{p}} \|\hat{\mu}\|_{L^p(L^2)} < \infty.$$

The $L^p(L^2)$ space is very close to the space $L^p(R^n)$. For instance, we have $\|f\|_{L^p(L^2)} \leq C\|f\|_{L^p(R^n)}$ for $p \geq 2$, and $\|f\|_{L^p(L^2)} = C\|f\|_{L^p(R^n)}$ if f is radial. The main result of this article is Theorem 4.1, which is stated in terms of $L^p(L^2)$ spaces. In particular, it yields that if $\text{supp } \mu \subset S^{n-1}$, then (3) is equivalent to (2). Since it is trivial to see that (1) implies (3), we conclude that if $\text{supp } \mu \subset S^{n-1}$, then (1) implies (2).

The well known optimal $L^p - L^2$ restriction theorem says that the Fourier transform restriction operator T is bounded from $L^p(R^n)$ to $L^2(S^{n-1})$ for all $1 \leq p \leq \frac{2(n+1)}{n+3}$. If one considers the $L^p - L^q$ restriction problem, then the optimal range of p should be $1 \leq p < \frac{2n}{n+1}$. This turns out to be a long-standing conjecture for $n \geq 3$ ([11], page 388). In [12], L. Vega proved the $L^p(L^2)(R^n) \rightarrow L^2(S^{n-1})$ restriction theorem for p in the optimal range $1 \leq p < \frac{2n}{n+1}$. As a corollary of Theorem 4.1, we will recapture L. Vega's result with the precise information for the constant in the inequality (see Remark 4.2).

Working with the mixed norm spaces enables us to substitute the Plancherel theorem with the Parseval identity for Fourier coefficients so that $p > 2$ in the r direction is allowed, while in the previous work, $p = 2$ in all directions is imperative. For our approach, the difficulty stems from the assumption that $p > \frac{2n}{n-1}$, since, when invoking the Hölder inequality in the Lorentz spaces (see [5]), one can only assert that if $\hat{\mu} \in L^{\frac{2n}{n-1}, \infty}$ (the weak $L^{\frac{2n}{n-1}}$ space), then (2) is satisfied. The key ingredients in our proof are the $L^p(L^2)$ version of Theorem 1.1 and the best possible

uniform L^p estimate of the Bessel functions $J_\nu(r)$ for all nonnegative real ν , which is proved via the Fourier transform restriction theorem on S^1 .

We organize this paper as follows. Section 2 supplies some background information. Section 3 is devoted to the estimates of Bessel functions. The main result is proved in Section 4, where a counter-example is also given to show that for any fixed $p > 4$, there exists a distribution $\mu \in S'(R^2)$ with $\text{supp } \mu \subset S^1$ and $\hat{\mu} \in L^p(L^2)$ such that μ is not a finite measure on S^1 .

2. PRELIMINARIES

We start with the dual form of the restriction theorem of the Fourier transform to the unit circle S^1 in R^2 . We refer the reader to [11], page 414, for the original form. In the rest of this paper, the same C will stand for different uniform constants.

Lemma 2.1. *Let $\mu = g(s)ds$ be a finite measure on S^1 . Then there is a constant C , independent of p , such that*

$$(4) \quad \left(\int_{R^2} |\hat{\mu}(\xi)|^p d\xi \right)^{\frac{1}{p}} \leq C(p-4)^{-\frac{1}{p}} \left(\int_{S^1} |g(s)|^q ds \right)^{\frac{1}{q}}, \quad \frac{3}{p} + \frac{1}{q} = 1, 4 < p < \infty.$$

Proof. For $0 < \alpha < n$ and $1 < s < 2$, the proof of the Hardy-Littlewood-Sobolev inequality ([11], page 354) indicates the inequality

$$(5) \quad \|f * |y|^{-\alpha}\|_{L^{s'}(R^n)} \leq C(n-\alpha)^{-1} \|f\|_{L^s(R^n)}, \quad \frac{2}{s} = 1 + \frac{n-\alpha}{n},$$

where $s' = \frac{s}{s-1}$ and C is independent of α (or s).

Then an application of the Hölder inequality yields

$$(6) \quad \left| \int_{R^n} \int_{R^n} f(x)f(y)|x-y|^{-\alpha} dx dy \right| \leq C(n-\alpha)^{-1} \|f\|_{L^s(R^n)}^2$$

with $0 < \alpha < n$ and $1 < s < 2$, such that $\frac{2}{s} = 1 + \frac{n-\alpha}{n}$.

Since S^1 is compact, we may cut it into several pieces if necessary so that for each piece we can follow the proof of Theorem 5.1 in [11], pages 412–414, to get

$$(7) \quad \left(\int_{R^2} |\hat{\mu}(\xi)|^{2r'} d\xi \right)^{\frac{r}{r'}} \leq A \int_{R^1} \int_{R^1} |f(x)|^r |f(y)|^r |x-y|^{1-r} dx dy,$$

where $1 < r < 2$ and $r' = \frac{r}{r-1}$. Here the constant A is independent of r .

Using (6) for $n = 1$ to control the right-hand side of (7), we have

$$(8) \quad \left(\int_{R^2} |\hat{\mu}(\xi)|^{2r'} d\xi \right)^{\frac{r}{r'}} \leq C(2-r)^{-1} \left(\int_{R^1} |f(x)|^{rs} dx \right)^{\frac{2}{s}},$$

where $1 < r < 2$, $r' = \frac{r}{r-1}$, and $\frac{2}{s} = 3 - r$.

Let $p = 2r'$ and $q = rs$; then an easy calculation shows that $\frac{3}{p} + \frac{1}{q} = 1$. Thus (4) follows from (8) since there is a constant C , independent of r (or p), such that $(2-r)^{-\frac{1}{2r}} \leq C(p-4)^{-\frac{1}{p}}$ for all $1 < r < 2$ and hence for all $4 < p < \infty$. The proof of Lemma 2.3 is complete

The following lemma is called van der Corput's lemma.

Lemma 2.2 ([11], page 332). *Suppose that ϕ is real-valued and smooth in (a, b) and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$. Then there exists a constant c_k independent of ϕ, t, a and b such that for $t > 0$*

$$\left| \int_a^b e^{it\phi(x)} dx \right| \leq c_k t^{-\frac{1}{k}}$$

holds when (i) $k \geq 2$, or (ii) $k = 1$ and $\phi'(x)$ is monotone.

For $x \in R^n$, write $x = rx'$, with $r = |x|$ and $x' \in S^{n-1}$. We will need the following identity, which can be derived easily from Theorem 3.10, page 158 in [10].

Lemma 2.3. *Let $\mu = Y(x') dx'$ with $Y(x')$ a spherical harmonic of degree $m (\geq 0)$. Then*

$$\hat{\mu}(\xi) = Ci^{-m} r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}+m}(r) Y(\xi'),$$

where C is independent of m .

Checking the proof of Theorem 1.1 in [2] carefully, one sees that the same proof with a minor modification yields its $L^p(L^2)$ version as follows.

Lemma 2.4. *If*

$$\begin{aligned} \hat{\mu} \in L^p(L^2), \quad & 4 < p \leq \infty, n = 2, \\ 3 < p < \infty, \quad n = 3, \quad & \frac{2n}{n-1} < p < \frac{2n}{n-3}, \quad n > 3, \end{aligned}$$

then (C) implies (B).

3. ESTIMATES OF BESSEL FUNCTIONS

Recall that for integers m , the Bessel functions $J_m(r)$ are defined by

$$J_m(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(r \sin \theta - m\theta)} d\theta$$

and we have $J_{-m}(r) = (-1)^m J_m(r)$. Also for positive real numbers v , the Bessel functions $J_v(r)$ can be written as $A_v(r) - B_v(r)$, where

$$(9) \quad A_v(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(r \sin \theta - v\theta)} d\theta,$$

$$(10) \quad B_v(r) = \frac{\sin(v\pi)}{\pi} \int_0^\infty e^{-vt - r \sinh(t)} dt.$$

We refer the readers to [13], page 176, for details.

Lemma 3.1. *For $0 < \lambda \leq \frac{1}{2}$, we have*

$$(11) \quad \int_1^\infty J_v^2(r) r^{-\lambda} dr \geq C \frac{1}{\lambda}, \quad 0 \leq v \leq 1,$$

$$(12) \quad \int_1^\infty J_v^2(r) r^{-\lambda} dr \geq C \frac{1}{\lambda} \cdot \frac{1}{v^\lambda}, \quad v \geq 1,$$

where C is independent of v, λ .

Proof. From the definition of $J_v(r)$, it is obvious that $|J_v(r)| \leq C$ for all $v \geq 0$ and $r \geq 0$. It follows that $\int_0^1 J_v^2(r)r^{-\lambda} dr \leq C$ for $0 < \lambda \leq \frac{1}{2}$ and $0 \leq v \leq 1$. Also from (9) and (10), we have $\int_0^1 J_v^2(r)r^{-\lambda} dr \leq C\frac{1}{v^2}$ for $0 < \lambda \leq \frac{1}{2}$ and $v \geq 1$. Thus the lemma will be proved if we can show that

$$(13) \quad \int_0^\infty J_v^2(r)r^{-\lambda} dr \geq C\frac{1}{\lambda}, \quad 0 < \lambda \leq \frac{1}{2}, \quad 0 \leq v \leq 1,$$

$$(14) \quad \int_0^\infty J_v^2(r)r^{-\lambda} dr \geq C\frac{1}{\lambda} \cdot \frac{1}{v^\lambda}, \quad 0 < \lambda \leq \frac{1}{2}, \quad v \geq 1.$$

The following formula can be found in [13], page 403, which we only need for $v \geq 0$ and $0 < \lambda \leq \frac{1}{2}$, even though it holds in other cases:

$$(15) \quad \int_0^\infty J_v^2(r)r^{-\lambda} dr = \frac{(\frac{1}{2})^{\lambda-1}\Gamma(\lambda)\Gamma(v - \frac{1}{2}\lambda + \frac{1}{2})}{2(\Gamma(\frac{1}{2}\lambda + \frac{1}{2}))^2\Gamma(v + \frac{1}{2}\lambda + \frac{1}{2})}.$$

The inequality (13) will follow from (15) and the fact that $\Gamma(\lambda) \geq C\frac{1}{\lambda}$ for $0 < \lambda \leq \frac{1}{2}$. The inequality (14) will follow from (15) and the following inequality whose proof is an easy exercise in calculus:

$$(16) \quad \frac{\Gamma(v - \frac{1}{2}\lambda + \frac{1}{2})}{\Gamma(v + \frac{1}{2}\lambda + \frac{1}{2})} \geq C\frac{1}{v^\lambda}, \quad 0 < \lambda \leq \frac{1}{2}, \quad v \geq 1.$$

The proof of the lemma is finished.

Lemmas 2.1 and 2.3 and the fact that $J_{-m}(r) = (-1)^m J_m(r)$ give the following uniform L^p estimate of Bessel functions $J_m(r)$ for all integers m . This estimate seems to be well known in the community, but it is not available for the author in the literature.

Lemma 3.2.

$$(17) \quad \left(\int_0^\infty |J_m(r)|^p r dr \right)^{\frac{1}{p}} \leq C(p-4)^{-\frac{1}{p}},$$

where C is independent of $p > 4$ and all integers m .

Remark. For each fixed m , Lemma 3.2 is trivial due to the fact that as $r \rightarrow \infty$, $J_m(r)$ is essentially $C_m r^{-\frac{1}{2}}$. What makes Lemma 3.2 interesting is that $C_m \rightarrow \infty$ as $m \rightarrow \infty$ ([11], page 357).

In the remainder of this section, we will prove that the same estimate (17) holds for $J_v(r)$, uniformly for all $v \geq 0$.

Lemma 3.3. For $v \geq 2$, we have

$$(18) \quad (i) \quad \left(\int_0^\infty |J_{v-1}(r) - J_{v+1}(r)|^p r dr \right)^{\frac{1}{p}} \leq C(p-4)^{-\frac{1}{p}},$$

$$(19) \quad (ii) \quad \left(\int_0^\infty |J_v(r) + J_{v+1}(r)|^p r dr \right)^{\frac{1}{p}} \leq C(p-4)^{-\frac{1}{p}}$$

where C is independent of $p > 4$ and v .

Proof. First of all it is easy to check that (17) is true for $B_v(r)$, uniformly for $v \geq 1$. Thus it remains to verify (18) for $A_{v-1}(r) - A_{v+1}(r)$ and (19) for $A_v(r) + A_{v+1}(r)$.

Write v as $m + \alpha$ with $0 \leq \alpha < 1$. Let $F_\alpha(\theta) = e^{i(1-\alpha)\theta} - e^{-i(1+\alpha)\theta}$ and $G_\alpha(\theta) = e^{-\alpha\theta} + e^{-i(1+\alpha)\theta}$. Since $F_\alpha(\pi) = G_\alpha(\pi) = F_\alpha(-\pi) = G_\alpha(-\pi) = 0$, and both $F_\alpha(\theta)$ and $G_\alpha(\theta)$ are smooth, integrating by parts yields that for $\theta \in [-\pi, \pi]$, one has $F_\alpha(\theta) = \sum_{-\infty}^\infty a_k e^{ik\theta}$ and $G_\alpha(\theta) = \sum_{-\infty}^\infty b_k e^{ik\theta}$, where $\sum_{-\infty}^\infty |a_k| \leq C$ and $\sum_{-\infty}^\infty |b_k| \leq C$. Here the bound C is independent of α .

We have

$$A_{v-1}(r) - A_{v+1}(r) = \frac{1}{2\pi} \int_{-\pi}^\pi e^{i(r \sin \theta - m\theta)} (e^{i(1-\alpha)\theta} - e^{-i(1+\alpha)\theta}) d\theta,$$

$$A_v(r) + A_{v+1}(r) = \frac{1}{2\pi} \int_{-\pi}^\pi e^{i(r \sin \theta - m\theta)} (e^{-i\alpha\theta} + e^{-i(1+\alpha)\theta}) d\theta.$$

Invoking Lemma 3.2 and Minkowski’s integral inequality finishes the proof of (18) and (19).

Lemma 3.4.

$$|J_v(r)| \leq C \frac{1}{v}, \quad 0 \leq r \leq \frac{v}{2}, \quad v \geq 1,$$

$$|J_v(r)| \leq Cr^{-\frac{1}{3}}, \quad r \geq 1, \quad v \geq 0.$$

Proof. The estimates of this lemma are trivial for $B_v(r)$, hence we only prove the lemma for $A_v(r)$. We write $A_v(r)$ as $A_{v_1}(r) + A_{v_2}(r)$, where

$$2\pi A_{v_1}(r) = \int_{-\pi}^{-\frac{3\pi}{4}} e^{i(r \sin \theta - v\theta)} d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{i(r \sin \theta - v\theta)} d\theta + \int_{\frac{3\pi}{4}}^\pi e^{i(r \sin \theta - v\theta)} d\theta,$$

$$2\pi A_{v_2}(r) = \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} e^{i(r \sin \theta - v\theta)} d\theta + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{i(r \sin \theta - v\theta)} d\theta.$$

When $r \leq \frac{v}{2}$, we use Lemma 2.4 with $k = 1$ for both $A_{v_1}(r)$ and $A_{v_2}(r)$ to get

$$(20) \quad |A_v(r)| \leq C \frac{1}{v}, \quad 0 \leq r \leq \frac{v}{2}, \quad v \geq 1.$$

When $r \geq 1$, we use Lemma 2.4 with $k = 3$ to control $A_{v_1}(r)$ and with $k = 2$ to control $A_{v_2}(r)$. This together with (20) completes the proof of the lemma.

Now we are in a position to prove the main result in this section.

Theorem 3.5. *For $p > 4$, we have*

$$(21) \quad (i) \quad \left(\int_0^\infty |J_v(r)|^p r dr \right)^{\frac{1}{p}} \leq C(p-4)^{-\frac{1}{p}}, \quad v \geq 0,$$

$$(22) \quad (ii) \quad \int_0^\infty |J_v(r)|^p r dr \leq C_{p,\delta} v^{-\frac{\delta}{3}}, \quad v \geq 1, \quad 0 < \delta < p-4,$$

where C is independent of p and v , while $C_{p,\delta}$ depends on p and δ , but is independent of v .

Proof. The estimate (22) follows directly from (21) and Lemma 3.4. It remains to verify (21). For each fixed v , (21) is trivial, so we may assume $v \geq 2$. We will use

the following recurrence formulae ([13], page 45):

$$(23) \quad J_{v-1}(r) - J_{v+1}(r) = 2J'_v(r),$$

$$(24) \quad rJ'_v(r) - vJ_v(r) = -rJ_{v+1}(r).$$

From (18) and (23), we see that $J'_v(r)$ satisfies the estimate (21), uniformly for $v \geq 2$. It follows from (24) that $\frac{v}{r}J_v(r) - J_{v+1}(r)$ satisfies the estimate (21). This together with (19) yields that $(1 + \frac{v}{r})J_v(r)$ satisfies the estimate (21). In particular, there exists a constant C independent of $p > 4$ and $v \geq 2$ such that

$$\left(\int_0^{3v} \left| \left(1 + \frac{v}{r}\right) J_v(r) \right|^p r dr \right)^{\frac{1}{p}} \leq C(p-4)^{-\frac{1}{p}},$$

which implies

$$(25) \quad \left(\int_0^{3v} |J_v(r)|^p r dr \right)^{\frac{1}{p}} \leq C(p-4)^{-\frac{1}{p}}.$$

Using the cutting of the integral as in the proof of Lemma 3.4 and invoking Lemma 2.4 with $k = 1$ or $k = 2$, one can verify that there exists a constant C independent of $p > 4$ and $v \geq 2$ such that

$$(26) \quad \left(\int_{3v}^\infty |A_v(r)|^p r dr \right)^{\frac{1}{p}} \leq C(p-4)^{-\frac{1}{p}}.$$

As in the proof of Lemma 3.3, (26) is trivial for $B_v(r)$, $v \geq 2$. It follows that (26) is true for $J_v(r)$. Putting (25), (26) together completes the proof of the theorem.

4. PROOF OF THE MAIN RESULT

Theorem 4.1. *Let μ be a tempered distribution supported on S^{n-1} . Then*

(i) *if μ is a L^2 density on S^{n-1} , then*

$$(27) \quad \sup_{p > \frac{2n}{n-1}} \left(p - \frac{2n}{n-1} \right)^{\frac{1}{p}} \|\hat{\mu}\|_{L^p(L^2)} \leq C \|\mu\|_{L^2(S^{n-1})};$$

(ii) *if $\liminf_{p \rightarrow (\frac{2n}{n-1})^+} (p - \frac{2n}{n-1})^{\frac{1}{p}} \|\hat{\mu}\|_{L^p(L^2)} < \infty$, then μ is a L^2 density on S^{n-1} such that*

$$(28) \quad \|\mu\|_{L^2(S^{n-1})} \leq C \liminf_{p \rightarrow (\frac{2n}{n-1})^+} \left(p - \frac{2n}{n-1} \right)^{\frac{1}{p}} \|\hat{\mu}\|_{L^p(L^2)}$$

where C only depends on the dimension n .

Proof of (i). Without loss of generality, we assume that $\|\mu\|_{L^2(S^{n-1})} = 1$. We can write $\mu = g(x') dx'$ with $g(x') = \sum_0^\infty c_m Y_m(x')$, where $\sum_0^\infty |c_m|^2 = 1$ and $Y_m(x')$ is some spherical harmonic of degree m with $\|Y_m\|_2 = 1$. From Lemma 2.3, we have

$$\hat{\mu}(r, x') = C \sum_0^\infty i^{-m} c_m r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}+m}(r) Y_m(x').$$

Let $p > \frac{2n}{n-1}$ so that $p \cdot \frac{2(n-1)}{4} > 4$. We apply (i) of Theorem 3.5 to obtain

$$\begin{aligned} \left(\int_0^\infty |J_{\frac{n-2}{2}+m}|^{p \cdot \frac{2(n-1)}{n}}(r) r dr \right)^{\frac{n}{2(n-1)p}} &\leq C \left(p \cdot \frac{2(n-1)}{n} - 4 \right)^{-\frac{n}{2(n-1)p}} \\ &\leq C \left(p - \frac{2n}{n-1} \right)^{-\frac{n}{2(n-1)p}}. \end{aligned}$$

Moreover an easy calculation shows

$$\left(\int_1^\infty r^{(-\frac{(n-2)p}{2} + n - 1 - \frac{n}{2(n-1)}) \frac{2(n-1)}{n-2}} dr \right)^{\frac{n-2}{2(n-1)}} \leq C \left(p - \frac{2n}{n-1} \right)^{-\frac{n-2}{2(n-1)}}.$$

Therefore, the following inequalities are true:

$$\begin{aligned} &\int_1^\infty \left(\sum_0^\infty |c_m|^2 J_{\frac{n-2}{2}+m}^2(r) \right)^{\frac{p}{2}} r^{-\frac{(n-2)p}{2}} r^{n-1} dr \\ &\leq C \int_1^\infty \left(\sum_0^\infty |c_m|^2 |J_{\frac{n-2}{2}+m}|^p(r) \right) r^{-\frac{(n-2)p}{2}} r^{n-1} dr \\ &\leq C \sum_0^\infty |c_m|^2 \int_1^\infty |J_{\frac{n-2}{2}+m}|^p(r) r^{-\frac{(n-2)p}{2}} r^{n-1} dr \\ &\leq C \sum_0^\infty |c_m|^2 \int_1^\infty |J_{\frac{n-2}{2}+m}|^p(r) r^{\frac{n}{2(n-1)}} r^{-\frac{n}{2(n-1)}} r^{-\frac{(n-2)p}{2}} r^{n-1} dr \\ &\leq C \sum_0^\infty |c_m|^2 \left[\int_1^\infty |J_{\frac{n-2}{2}+m}|^{p \cdot \frac{2(n-1)}{n}}(r) r dr \right]^{\frac{n}{2(n-1)}} \\ &\quad \cdot \left[\int_1^\infty r^{(-\frac{(n-2)p}{2} + n - 1 - \frac{n}{2(n-1)}) \frac{2(n-1)}{n-2}} dr \right]^{\frac{n-2}{2(n-1)}} \\ &\leq C^p \left(p - \frac{2n}{n-1} \right)^{-\frac{n}{2(n-1)}} \left(p - \frac{2n}{n-1} \right)^{-\frac{n-2}{2(n-1)}} \\ &\leq C^p \left(p - \frac{2n}{n-1} \right)^{-1} \end{aligned}$$

where, in the first inequality, we used the Jensen convex inequality for the discrete integral since $\sum_0^\infty |c_m|^2 = 1$.

Since $\|Y_m\|_2 = 1$, which implies $\|Y_m\|_1 \leq C$ for all m , Lemma 2.3 implies that $|J_{\frac{n-2}{2}+m}(r)| r^{-\frac{(n-2)}{2}}$ are uniformly bounded by a constant B for all r and m .

This gives

$$\begin{aligned} &\int_0^1 \left(\sum_0^\infty |c_m|^2 J_{\frac{n-2}{2}+m}^2(r) \right)^{\frac{p}{2}} r^{-\frac{(n-2)p}{2}} r^{n-1} dr \\ &\leq B^p \sum_0^\infty |c_m|^2 \int_0^1 r^{n-1} dr \leq B^p. \end{aligned}$$

Thus the definition of $L^p(L^2)$ yields

$$\begin{aligned} \|\hat{\mu}\|_{L^p(L^2)}^p &= \int_0^\infty \left(\sum_0^\infty |c_m|^2 J_{\frac{n-2}{2}+m}^2(r) \right)^{\frac{p}{2}} r^{-\frac{(n-2)p}{2}} r^{n-1} dr \\ &= \left(\int_0^1 + \int_1^\infty \right) \left(\sum_0^\infty |c_m|^2 J_{\frac{n-2}{2}+m}^2(r) \right)^{\frac{p}{2}} r^{-\frac{(n-2)p}{2}} r^{n-1} dr \\ &\leq C^p \left(p - \frac{2n}{n-1} \right)^{-1} + B^p \end{aligned}$$

where C only depends on the dimension n . Since $\lim_{p \rightarrow \infty} \left(p - \frac{2n}{n-1} \right)^{\frac{1}{p}} = 1$, (27) follows from the last inequality.

Proof of (ii). Since S^{n-1} is compact, we see that $\langle \mu, f \rangle$ is well defined for $f \in C^\infty(R^n)$ and that $\hat{\mu}$ is a C^∞ function in R^n . In particular for each r , $\hat{\mu}(r, \xi')$ is a L^2 function on S^{n-1} . Under the assumption of (ii), we can apply Lemma 2.4 to see that $\langle \mu, f \rangle = 0$ if $f(x) \in C^\infty(R^n)$, vanishing on S^{n-1} . Thus we can write $\langle \mu, f \rangle = \langle \mu, f|_{S^{n-1}} \rangle$ for every $f \in C^\infty(R^n)$, where $f|_{S^{n-1}}$ is the restriction of f on S^{n-1} . In particular this implies that $\hat{\mu}(r, \xi') = \langle \mu, e^{ir\xi' \cdot x'} \rangle$ and for any spherical harmonic $Y(\xi')$ of degree m we have $\langle \mu, Y(\xi') \rangle = \langle \mu, P(\xi) \rangle$, where $P(\xi) = |\xi|^m Y\left(\frac{\xi}{|\xi|}\right)$ is a solid spherical harmonic of degree m . Now we have

$$\begin{aligned} &\int_{S^{n-1}} \hat{\mu}(r, \xi') Y(\xi') d\xi' \\ &= \int_{S^{n-1}} \langle \mu, e^{ir\xi' \cdot x'} \rangle Y(\xi') d\xi' \\ &= \left\langle \mu, \int_{S^{n-1}} e^{ir\xi' \cdot x'} Y(\xi') d\xi' \right\rangle \\ &= \langle \mu, C i^{-m} r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}+m}(r) Y(x') \rangle \\ &= C i^{-m} \langle \mu, Y(x') \rangle r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}+m}(r). \end{aligned}$$

Let $Y_{m_j}(\xi')$, $j = 1, 2, \dots, m_d$, be all of the spherical harmonics of degree m with $\|Y_{m_j}\|_2 = 1$. Let $c_{m_j} = \langle \mu, Y_{m_j} \rangle$, $c_m = \left(\sum_1^{m_d} |c_{m_j}|^2 \right)^{\frac{1}{2}}$ and

$$Y_m(\xi') = \frac{1}{c_m} \sum_1^{m_d} c_{m_j} Y_{m_j}(\xi')$$

provided $c_m \neq 0$.

With the above notation, we have shown that

$$\hat{\mu}(r, \xi') = C \sum i^{-m} c_m r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}+m} Y_m(\xi').$$

Given $p > \frac{2n}{n-1}$, let $\lambda_p = \left(\frac{n-1}{p} + 1 - \frac{2}{p} \right) \left(p - \frac{2n}{n-1} \right)$ so that

$$\left(n - 2 - \lambda_p + 1 + p - \frac{2n}{n-1} \right) \frac{p}{2} - \left(1 + p - \frac{2n}{n-1} \right) = n - 1$$

and that $0 < \lambda_p \leq C^2 \left(p - \frac{2n}{n-1} \right)$ with C independent of p . We may assume $\lambda_p \leq \frac{1}{2}$ by restricting p close to $\frac{2n}{n-1}$.

We have

$$\begin{aligned}
 & \left[\left(p - \frac{2n}{n-1} \right)^{-1} \left(|c_0|^2 + \sum_{m>0} \frac{|c_m|^2}{|m + \frac{n-2}{2}|^{\lambda_p}} \right) \right]^{\frac{p}{2}} \\
 & \leq \left[C^2 (\lambda_p)^{-1} \left(|c_0|^2 + \sum_{m>0} \frac{|c_m|^2}{|m + \frac{n-2}{2}|^{\lambda_p}} \right) \right]^{\frac{p}{2}} \\
 & \leq C^p \left[\sum_{m \geq 0} |c_m|^2 \int_1^\infty J_{m + \frac{n-2}{2}}^2(r) r^{-\lambda_p} dr \right]^{\frac{p}{2}} \\
 & \leq C^p \left[\int_1^\infty \sum_{m \geq 0} |c_m|^2 J_{m + \frac{n-2}{2}}^2(r) r^{-(n-2)} r^{(n-2)} r^{-\lambda_p} r^{1+p-\frac{2n}{n-1}} r^{-(1+p-\frac{2n}{n-1})} dr \right]^{\frac{p}{2}} \\
 & \leq C^p \left(p - \frac{2n}{n-1} \right)^{1-\frac{p}{2}} \int_1^\infty \left(\sum_{m \geq 0} |c_m|^2 J_{m + \frac{n-2}{2}}^2(r) r^{-(n-2)} \right. \\
 & \quad \left. \cdot (r^{(n-2)} r^{-\lambda_p} r^{1+p-\frac{2n}{n-1}})^{\frac{p}{2}} r^{-(1+p-\frac{2n}{n-1})} dr \right)^{\frac{p}{2}} \\
 & \leq C^p \left(p - \frac{2n}{n-1} \right)^{1-\frac{p}{2}} \int_0^\infty \left(\int_{S^{n-1}} |\hat{\mu}(r, x')|^2 dx' \right)^{\frac{p}{2}} r^{n-1} dr \\
 & \leq C^p \left(p - \frac{2n}{n-1} \right)^{-\frac{p}{2}} \left(p - \frac{2n}{n-1} \right) \|\hat{\mu}\|_{L^p(L^2)}^p.
 \end{aligned}$$

Here the second inequality comes from Lemma 3.1 and, in the fourth inequality, we used Jensen’s convex inequality for integrals and the fact that $\int_1^\infty r^{-(1+p-\frac{2n}{n-1})} dr = (p - \frac{2n}{n-1})^{-1}$.

Thus we have

$$\left(|c_0|^2 + \sum_{m>0} \frac{|c_m|^2}{|m + \frac{n-2}{2}|^{\lambda_p}} \right)^{\frac{1}{2}} \leq C \left(p - \frac{2n}{n-1} \right)^{\frac{1}{p}} \|\hat{\mu}\|_{L^p(L^2)},$$

where C only depends on the dimension n . Since $\lambda_{p_k} \rightarrow 0$ for any $p_k \rightarrow (\frac{2n}{n-1})^+$, we get

$$\left(\sum_{m \geq 0} |c_m|^2 \right)^{\frac{1}{2}} \leq C \liminf_{p \rightarrow (\frac{2n}{n-1})^+} \left(p - \frac{2n}{n-1} \right)^{\frac{1}{p}} \|\hat{\mu}\|_{L^p(L^2)},$$

which proves (28).

Remark 4.2. For $f \in S(R^n)$, the dual form of (27) is the following optimal mixed norm restriction result:

$$(29) \quad \|Tf\|_{L^2(S^{n-1})} \leq C \left(p' - \frac{2n}{n-1} \right)^{-\frac{1}{p'}} \|f\|_{L^p(L^2)}, \quad 1 \leq p < \frac{2n}{n-1},$$

where p' is given by $\frac{1}{p'} = 1 - \frac{1}{p}$ and C is independent of p .

We now turn our attention to the counter-example. We start with a lemma that is just the Fourier series version of Theorem 7.6.6, page 209 in [4].

Lemma 4.3. For a small $\varepsilon > 0$ and any $p > 2$, there exists $\mu \in S'(R)$ with $\text{supp } \mu \subset [\varepsilon, 2\pi - \varepsilon]$, not a finite measure, such that $\sum_{-\infty}^{\infty} |\hat{\mu}(m)|^p < \infty$, where $c_m = \langle \mu, e^{imx} \rangle$.

Proof. Checking the proof in [4] for the case $n = 1$, one can see that everything could be adopted except the fact that the Fourier transform of $e^{it\xi^2}$ is $Ct^{-\frac{1}{2}}e^{-it\frac{\xi^2}{4t}}$, which was used to show that the Fourier transform of $e^{it\xi^2}$ is uniformly bounded by $Ct^{-\frac{1}{2}}$. The author is not aware of the exact formula for the Fourier coefficients of the function $e^{it\xi^2}$ on $[0, 2\pi]$. Fortunately, by invoking van de Corput's lemma for $k = 2$, we can have the same estimate that the Fourier coefficient of $e^{it\xi^2}$ on $[0, 2\pi]$ is uniformly bounded by $Ct^{-\frac{1}{2}}$.

Example 4.4. For any $p > 4$, there exists a tempered distribution μ supported on S^1 , not a finite measure, such that $\hat{\mu} \in L^p(L^2)$.

Proof. Since $p > 4$, we have

$$\frac{4(7-p)}{p+8} < \frac{4}{p}.$$

Choose α such that

$$\frac{4(7-p)}{p+8} < \alpha < \frac{4}{p}.$$

From $\frac{4(7-p)}{p+8} < \alpha$, we have

$$\frac{4-\alpha p}{4-2\alpha} < \frac{p-4}{3}.$$

Let $l = \frac{4-2\alpha}{4-\alpha p}$; then $l > 1$ since $p > 4 > 2$. From the previous inequality, we have

$$\frac{p-4}{3} \cdot l > 1.$$

Let l' be the dual of l ; then

$$\frac{\alpha p}{2} \cdot l' = (2-\alpha) \frac{p}{p-2}.$$

From $\alpha < \frac{4}{p}$, we see that

$$(2-\alpha) \frac{p}{p-2} > 2.$$

Now from Lemma 4.3, we can find a tempered distribution $\mu \in S'(R)$ with $\text{supp } \mu \subset [\varepsilon, 2\pi - \varepsilon]$, not a finite measure, such that $\sum_{-\infty}^{\infty} |c_m|^{(2-\alpha)\frac{p}{p-2}} < \infty$. Notice that the same μ can be viewed as a tempered distribution in R^2 with $\text{supp } \mu \subset S^1$, but not a finite measure on S^1 . Since $\frac{p-4}{3}l > 1$, (ii) of Theorem 3.5 yields

$$\sum_{-\infty}^{\infty} \left(\int_0^{\infty} |J_m|^p(r)r \, dr \right)^l < \infty.$$

The following inequalities will finish the proof of the example:

$$\begin{aligned}
\|\hat{\mu}\|_{L^p(L^2)}^p &= C \int_0^\infty \left(\sum_{-\infty}^\infty |c_m|^2 J_m^2(r) \right)^{\frac{p}{2}} r dr \\
&= C \int_0^\infty \left(\sum_{-\infty}^\infty |c_m|^{2-\alpha} \cdot |c_m|^\alpha J_m^2(r) \right)^{\frac{p}{2}} r dr \\
&\leq C \int_0^\infty \left(\sum_{-\infty}^\infty |c_m|^{(2-\alpha)\frac{p}{p-2}} \right)^{\frac{p}{2} \cdot \frac{p-2}{p}} \left(\sum_{-\infty}^\infty |c_m|^{\frac{\alpha p}{2}} |J_m|^p(r) \right) r dr \\
&\leq C \sum_{-\infty}^\infty |c_m|^{\frac{\alpha p}{2}} \int_0^\infty |J_m|^p(r) dr \\
&\leq C \left(\sum_{-\infty}^\infty |c_m|^{\frac{\alpha p}{2} \cdot l'} \right)^{\frac{1}{l'}} \left(\sum_{-\infty}^\infty \left(\int_0^\infty |J_m|^p(r) r dr \right)^{l'} \right)^{\frac{1}{l'}} \\
&< \infty.
\end{aligned}$$

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