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EXPLICIT FREE SUBGROUPS OF Aut($\mathbf{R}, \leq$)

CURTIS D. BENNETT

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ABSTRACT. In this note for any finite $n > 1$, we give an explicit free subgroup of rank n of the groups of ordered permutations of the reals (Aut($\mathbf{R}, \leq$)) for which the proof that the subgroup is free is elementary. Moreover, this example naturally generalizes to the group Aut($\mathbf{Q}, \leq$).

1. INTRODUCTION

It has long been known that the group of order-preserving permutations of the reals ($A(\mathbf{R}) = \mathrm{Aut}(\mathbf{R}, \leq)$) contains free subgroups of rank $2^{\aleph_0}$. Currently, there is no simple explicit construction of a free subgroup of rank greater than one, for which the freeness is easy to prove. The purpose of this paper is to provide such an explicit example.

We will briefly discuss some of the known proofs of the existence of free subgroups of $A(\mathbf{R})$, starting from the least constructive and ending with the most natural. In 1949, B. H. Neumann showed that any free group can be totally ordered so that the group operation preserves the order. Moreover if the rank is at most $2^{\aleph_0}$, this ordering can be done so that the ordered set of the group elements is embeddable in $\mathbf{R}$. From this it is easy to see that free groups can be embedded in $A(\mathbf{R})$. Of course, this proof is not constructive. Furthermore, the proof requires several difficult theorems.

A more constructive proof can be obtained using a patching argument (see [GM] for example). In this argument, which we will outline in the case of two generators, we first order all nontrivial words $w(a,b)$ on the generators a and b. For the ith word $w(a,b)$, we concoct a pair $g_i^{(1)}, g_i^{(2)}$ of ordered permutations of the interval $[i, i+1]$ such that $w(g_i^{(1)}, g_i^{(2)})$ is not the identity on $[i, i+1]$. For $j \in \{1, 2\}$, then define

$$g^{(j)}(x) = \begin{cases} x & \text{if } x < 1, \\ g^{(j)}_{\lfloor x \rfloor}(x) & \text{if } x \geq 1, \end{cases}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. It follows that $g^{(1)}$ and $g^{(2)}$ generate a free subgroup of $A(\mathbf{R})$. This approach is extremely useful for constructing free subgroups which are highly transitive. Unfortunately, the generators are neither natural nor explicitly stated.

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Perhaps the most natural free subgroup on two generators is given by White's Theorem ([W]). In that paper, S. White shows that for any odd prime p, the group generated by the functions $f(x) = x + 1$ and $g(x) = x^p$ is free. This is a very natural and beautiful result and surprisingly difficult to prove. White's proof uses Galois closures and other results from field theory.

While the functions that we give are not as natural as those in White's theorem, it is significantly easier to prove that they generate a free subgroup of $A(\mathbf{R})$. Since our functions are also elements of $A(\mathbf{Q})$ ($= \mathrm{Aut}(\mathbf{Q}, \leq)$) if one restricts the domain, they also give a free subgroup of $A(\mathbf{Q})$. Whereas a result of J. D. Dixon [D] can be modified to $A(\mathbf{Q})$ to imply that this group contains free subgroups (e.g., see [GMR], where it is shown that almost all finitely generated subgroups are free, where "almost all" is properly interpreted); to our knowledge, nice explicit examples for $A(\mathbf{Q})$ do not exist.

At this point, we would like to express our thanks to the referee for his many helpful suggestions.

2. Construction and proof

In this section, for a given integer $n > 1$, we will construct n elements of $A(\mathbf{R})$ which will be the free generators of a free subgroup. Throughout, for $a \in \mathbf{R}$, we define $T_a\colon \mathbf{R} \to \mathbf{R}$ by $T_a(x) = x - a$.

Let

$$f_n(x) = \begin{cases} 2nx, & x \in [0, \frac{1}{2n+1}], \\ \frac{2n}{2n+1} + \frac{1}{2n}\left(x - \frac{1}{2n+1}\right), & x \in [\frac{1}{2n+1}, 1]. \end{cases}$$

Then let $\overline{f}_n\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$\overline{f}_n(y) = \lfloor y \rfloor + f_n(y - \lfloor y \rfloor) = (T_{\lfloor y \rfloor})^{-1} \circ f_n \circ T_{\lfloor y \rfloor}(y),$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

We now note a few easy facts about the function f_n which maps $[0,1]$ onto $[0,1]$ in an order-preserving fashion.

Lemma 2.1. *Let $f_n(x)$ be defined as above.*

1. *If $x \in [\frac{1}{2n+1}, 1]$ and $l > 0$, then $(f_n)^l(x) \in [\frac{2n}{2n+1}, 1]$.*
2. *If $x \in [0, \frac{2n}{2n+1}]$ and $l < 0$, then $(f_n)^l(x) \in [0, \frac{1}{2n+1}]$.*

Proof. The first conclusion is immediate from the definition of f_n and induction. For the second, note that $f_n^{-1}(x) = \frac{1}{2n}x$ for $x \in [0, \frac{2n}{2n+1}]$. The result now follows. □

We now define our free generators. For $i \in \{0, \dots, n-1\}$, let

$$g_{(i,n)} = (T_{\frac{i}{n}})^{-1} \circ \overline{f}_n \circ T_{\frac{i}{n}}.$$

Theorem 2.2. *For n a positive integer greater than 1, the set*

$$\{g_{(0,n)}, \dots, g_{(n-1,n)}\} \subset A(\mathbf{R})$$

generates a free subgroup of rank n.

Proof. Let

$$\Omega_i = \bigcup_{k \in \mathbf{Z}} \left[k + \frac{i}{n} - \frac{1}{2n+1}, k + \frac{i}{n} + \frac{1}{2n+1}\right].$$

It is useful to note that $\Omega_i = (T_{\frac{i}{n}})^{-1}(\Omega_0)$.

We first claim that if $l \neq 0$ and $x \notin \Omega_i$, then $(g_{(i,n)})^l(x) \in \Omega_i$. To see this, note that

$$(g_{(i,n)})^l(x) = ((T_{\frac{i}{n}})^{-1} \circ (\overline{f_n})^l \circ T_{\frac{i}{n}})(x),$$

and $x \notin \Omega_i$ implies that $T_{\frac{i}{n}}(x) \in (k + \frac{1}{2n+1}, k + \frac{2n}{2n+1})$ for some integer k. Hence by Lemma 2.1

$$(g_{(i,n)})^l(x) \in (T_{\frac{i}{n}})^{-1}\left(\left[k + \frac{2n}{2n+1}, k+1\right]\right) \subset \Omega_i \tag{1}$$

for $l > 0$, and

$$(g_{(i,n)})^l(x) \in (T_{\frac{i}{n}})^{-1}\left(\left[k, k + \frac{1}{2n+1}\right]\right) \subset \Omega_i \tag{2}$$

for $l < 0$.

Our second claim is that the Ω_i's are disjoint. To see this, assume without loss of generality that there exists $z \in \Omega_0 \cap \Omega_i$ for some $i \in \{1, \ldots, n-1\}$. That is, we have $x, y \in [-\frac{1}{2n+1}, \frac{1}{2n+1}]$ such that

$$x + k = y + l + \frac{i}{n}$$

for some integers k, l, and i with $0 < i < n$. Reducing the above, we see that

$$x - y = m + \frac{i}{n}$$

for some integers m and i. Hence $m + \frac{i}{n} \in [-\frac{2}{2n+1}, \frac{2}{2n+1}]$. Since $n > 1$ and $\frac{1}{n} > \frac{2}{2n+1}$, we must have $m = i = 0$. This contradicts our choice of i. Hence the Ω_i's are disjoint.

To finish the proof of the theorem we begin by choosing any $x \notin \bigcup_{i=0}^{n-1} \Omega_i$. For example, $\frac{1}{2n}$ is such an x as it lies in the interval $(\frac{1}{2n+1}, \frac{1}{n} - \frac{1}{2n+1})$. Examine

$$(g_{(i_1,n)})^{l_1}(g_{(i_2,n)})^{l_2} \cdots (g_{(i_k,n)})^{l_k}(x),$$

where we assume $i_j \neq i_{j+1}$ and $l_j \neq 0$ for $j \in \{1, \ldots, k\}$. Proceed by induction. Since $x \notin \Omega_{i_k}$, equations (1) and (2) imply $(g_{(i_k,n)})^{l_k}(x) \in \Omega_{i_k}$. Now, suppose $j > 1$ and

$$x_j = (g_{(i_j,n)})^{l_j} \cdots (g_{(i_k,n)})^{l_k}(x) \in \Omega_{i_j}.$$

Since $\Omega_{i_{j-1}} \cap \Omega_{i_j} = \varnothing$, equations (1) and (2) imply $x_{j-1} = (g_{(j-1,n)})^{l_{j-1}}(x_j) \in \Omega_{i_{j-1}}$. Hence,

$$x_1 = (g_{(i_1,n)})^{l_1}(g_{(i_2,n)})^{l_2} \cdots (g_{(i_k,n)})^{l_k}(x) \in \Omega_{i_1}.$$

Since $x \notin \Omega_{i_1}$, we therefore have $x \neq x_1$. As a result, no word in the $g_{(i,n)}$'s can be the identity function. This is, of course, what it means for the set $\{g_{(0,n)}, \ldots, g_{(n-1,n)}\}$ to generate a free subgroup of rank n. □

The technique used at the end of the proof is a very specific application of a general theorem of Macbeath (see [M] and [LS, Section III.12] for the more general theorem). In turn, the result of Macbeath is a generalization of Klein's work in the case of discrete groups [K].

Note that each $g_{(i,n)}$ is clearly a bijection on the rationals. Hence, viewing $g_{(i,n)}$ as an ordered permutation of $\mathbf{Q}$, the result holds for $A(\mathbf{Q})$. Another feature of these generators is that the function $g_{(i,n)}(x) - x$ is a periodic function on $\mathbf{R}$ in the trigonometric sense.

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Department of Mathematics, Bowling Green State University, Bowling Green, Ohio 43403

E-mail address: cbennet@math.bgsu.edu