

## THE STRUCTURE OF HYPERSURFACES WITH SOME CURVATURE CONDITIONS

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ABSTRACT. Let  $M$  be a hypersurface in  $\mathbf{R}^{n+1}$ , and let  $H, R$  denote the mean curvature and the scalar curvature of  $M$  respectively. We show that if  $M$  is compact and  $R > \frac{n-2}{n-1}H^2$ , then  $M$  is diffeomorphic to  $S^n$ . Also we prove that if  $M$  is complete,  $H$  is constant and  $R \geq \frac{n-2}{n-1}H^2$ , then  $M$  is  $\mathbf{R}^n$  or  $S^n$  or  $S^{n-1} \times \mathbf{R}^1$ .

### 1. INTRODUCTION AND PRELIMINARIES

If  $M$  is a hypersurface in Euclidean space, then its sectional curvature, Ricci curvature, scalar curvature, mean curvature, and principal curvatures are defined at each point, are related to each other, and are important ingredients of the structure of  $M$ .

S. Y. Cheng and S. T. Yau showed in [3] that a complete hypersurface in  $\mathbf{R}^{n+1}$  with constant mean curvature and nonnegative sectional curvature is a generalized cylinder  $S^{n-k} \times \mathbf{R}^k$ . On the other hand, Q. M. Cheng and H. Nakagawa showed the following in [2]: Let  $M$  be a complete hypersurface with constant mean curvature of an  $(n+1)$ -dimensional space form  $M^{n+1}(c)$ ,  $c \geq 0$ . If the length  $S$  of the second fundamental form satisfies

$$\sup S^2 < [n\{2(n-1)c + H^2\} - (n-2)|H|\{H^2 + 4(n-1)c\}^{1/2}]/2(n-1)$$

where  $H$  is the trace of the second fundamental form, then  $M$  is totally umbilical. This theorem says, in case of  $c = 0$ , the following:

**Proposition.** *If  $M$  is a complete hypersurface with constant mean curvature embedded in  $\mathbf{R}^{n+1}$  and if the scalar curvature  $R$  satisfies  $R > \frac{n-2}{n-1}H^2$ , then  $M$  is totally umbilical.*

In this paper, we prove that if  $M$  is a compact hypersurface and  $R > \frac{n-2}{n-1}H^2$ , then  $M$  is diffeomorphic to  $S^n$ , using a lemma by K. R. Cai. We also prove that if  $M$  is a complete hypersurface with constant mean curvature and  $R \geq \frac{n-2}{n-1}H^2$ , then  $M$  is the plane  $\mathbf{R}^n$  or the sphere  $S^n$  or the cylinder  $S^{n-1} \times \mathbf{R}^1$ , which gives another simple proof of the above proposition when  $R > \frac{n-2}{n-1}H^2$ .

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Throughout this paper,  $M^n$  or  $M$  denotes an  $n$ -dimensional connected hypersurface embedded in  $\mathbf{R}^{n+1}$ . We denote the mean curvature (the trace of the second fundamental form) by  $H$ . We summarize the results in [4] used in this paper.

Fix  $p \in M$ . Let  $\{e_i\}_{1 \leq i \leq n}, \{\lambda_i\}_{1 \leq i \leq n}$  be the principal vectors and the corresponding principal curvatures, respectively, at  $p$ . Let  $x, y$  be unit vectors with  $x \perp y$  and  $x = \sum_{i=1}^n a_i e_i, y = \sum_{j=1}^n b_j e_j$ . Then the sectional curvature  $K$  determined by  $x, y$  is given by

$$\begin{aligned}
 K(x, y) &= \langle A(x, x), A(y, y) \rangle - \langle A(x, y), A(x, y) \rangle \\
 (1.1) \qquad &= \left( \sum_{i=1}^n a_i^2 \lambda_i \right) \left( \sum_{j=1}^n b_j^2 \lambda_j \right) - \left( \sum_{i=1}^n a_i b_i \lambda_i \right)^2 \\
 &= \sum_{i < j} (a_i b_j - a_j b_i)^2 \lambda_i \lambda_j,
 \end{aligned}$$

where  $A$  is the second fundamental form of  $M$ . Since  $\{x, y\}$  are orthonormal vectors, we have

$$\begin{aligned}
 1 &= (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2) - (a_1 b_1 + \dots + a_n b_n)^2 \\
 &= \sum_{i \neq j} a_i^2 b_j^2 - 2 \sum_{i < j} a_i b_i a_j b_j \\
 (1.2) \qquad &= \sum_{i < j} (a_i b_j)^2 - 2 \sum_{i < j} a_i b_j a_j b_i + \sum_{i < j} (a_j b_i)^2 \\
 &= \sum_{i < j} (a_i b_j - a_j b_i)^2.
 \end{aligned}$$

Hence we have

$$(1.3) \qquad \min_{i \neq j} \lambda_i \lambda_j \leq K(x, y) \leq \max_{i \neq j} \lambda_i \lambda_j.$$

### 2. MAIN THEOREMS

K. R. Cai gives the following in [1]. We present its proof for the reader's convenience.

**Lemma 2.1.** *Let  $A = (a_{ij})$  be a symmetric  $n \times n$  matrix. Put  $A_1 = \text{tr}A$  and  $A_2 = \sum_{i,j} (a_{ij})^2$ . Then we have*

$$\begin{aligned}
 &\sum_i (a_{in})^2 - A_1 a_{nn} \\
 &\leq \left[ n(n-1)A_2 + (n-2)|A_1| \sqrt{(n-1)(nA_2 - A_1^2)} - 2(n-1)A_1^2 \right] / n^2.
 \end{aligned}$$

*Proof.* Since it holds trivially if  $n = 2$ , we assume that  $n > 2$ . Using the Lagrange multiplier method, we find an upper bound of the function  $-A_1 x_{nn} + \sum_i (x_{in})^2$  under the conditions

$$\sum_i x_{ii} = A_1, \quad \sum_i (x_{ii})^2 + 2 \sum_{i < j} (x_{ij})^2 = A_2.$$

Consider the function

$$f = \sum_i (x_{in})^2 - A_1 x_{nn} + \lambda \left( \sum_i x_{ii} - A_1 \right) + \mu \left[ \sum_i (x_{ii})^2 + 2 \sum_{i < j} (x_{ij})^2 - A_2 \right],$$

where  $\lambda, \mu$  are Lagrange multipliers. By differentiating with respect to  $\{x_{ij}\}$ , we obtain the following system of equations for critical points:

$$\begin{aligned} (2.1) \quad & \lambda + 2\mu x_{jj} = 0, \quad j = 1, \dots, n-1, \\ (2.2) \quad & \lambda - A_1 + 2(1 + \mu)x_{nn} = 0, \\ (2.3) \quad & 4\mu x_{ij} = 0, \quad i < j < n, \\ (2.4) \quad & (1 + 2\mu)x_{jn} = 0, \quad j < n. \end{aligned}$$

Then we obtain after some computation

$$(2.5) \quad -A_1 x_{nn} + 2 \sum_i (x_{in})^2 = -(\lambda A_1 + 2\mu A_2).$$

Without loss of generality, we may assume that  $\mu \neq 0$ . Then by (2.1) and (2.2),

$$(2.6) \quad 2\mu(x_{nn} - A_1) = (n - 1)\lambda.$$

By (2.2) and (2.6), we have  $2\mu(x_{nn} - A_1 + (n - 1)x_{nn}) = (n - 1)(A_1 - 2x_{nn})$ . Hence

$$(2.7) \quad \mu = \frac{(n - 1)(A_1 - 2x_{nn})}{2(n x_{nn} - A_1)}, \quad \frac{\lambda}{\mu} = \frac{2}{n - 1}(x_{nn} - A_1).$$

Suppose  $\mu \neq -\frac{1}{2}$ ; then by (2.4),  $x_{jn} = 0, 1 \leq j < n$ . Then by (2.5), we have

$$(2.8) \quad \frac{x_{nn}(A_1 - 2x_{nn})}{\mu} = \frac{\lambda}{\mu} A_1 + 2A_2.$$

Then we have, by (2.7) and (2.8),  $x_{nn}(n x_{nn} - A_1) = (x_{nn} - A_1)A_1 + A_2(n - 1)$ . Hence  $x_{nn} = \frac{A_1}{n} \pm \sqrt{\frac{n - 1}{n}(A_2 - \frac{A_1^2}{n})}$ . Then since  $x_{jn} = 0, j < n$ , we have two critical values of  $f = -A_1 x_{nn} + \sum_i (x_{in})^2$ :

$$\left[ n(n - 1)A_2 + (n - 2)|A_1| \sqrt{(n - 1)(nA_2 - A_1^2)} - 2(n - 1)A_1^2 \right] / n^2.$$

If  $\mu = -\frac{1}{2}$ , then by (1),  $\lambda = x_{ii} = 0$  and  $x_{nn} = A_1$ . Hence  $f = \frac{A_2}{2} - \frac{A_1^2}{2}$ . Therefore the assertion follows. □

Using this lemma, we prove a sphere theorem for a hypersurface.

**Theorem 2.2.** *Let  $M$  be a compact hypersurface of  $\mathbf{R}^{n+1}$ . If the mean curvature  $H$  and the scalar curvature  $R$  satisfy  $R > \frac{n - 2}{n - 1}H^2$ , then  $M$  is diffeomorphic to  $S^n$ .*

*Proof.* Let  $p \in M$ . Let  $\{e_1, \dots, e_n\}$  be the principal vectors and let  $\{\lambda_1, \dots, \lambda_n\}$  be the principal curvatures at  $p$ . Then we have

$$\text{Ric}(e_n, e_n) = \lambda_n(H - \lambda_n) = Hh_{nn} - \sum_i (h_{in})^2,$$

where  $(h_{ij})$  is the second fundamental form of  $M$ . Then at  $p$  we have

$$h_2 = \sum_{ij} (h_{ij})^2 = \sum_i h_{ii}^2 = \sum_i \lambda_i^2 = H^2 - R.$$

According to Lemma 2.1, at  $p$  we have

$$\begin{aligned} & \text{Ric}(e_n, e_n) \\ & \geq \frac{1}{n^2} \left[ -n(n-1)h_2 - (n-2)|H| \sqrt{(n-1)(nh_2 - H^2)} + 2(n-1)H^2 \right] \\ & = \frac{n-1}{n^2} \left[ -n(H^2 - R) - (n-2) \sqrt{H^2(H^2 - \frac{n}{n-1}R)} + 2H^2 \right] \\ & = -\frac{n-1}{n^2} \left[ (n-2)H^2 - nR + (n-2) \sqrt{H^2(H^2 - \frac{n}{n-1}R)} \right]. \end{aligned}$$

Since  $R > \frac{n-2}{n-1}H^2, nR - (n-2)H^2 > 0$ . Moreover,

$$\begin{aligned} & \left[ (n-2) \sqrt{H^2(H^2 - \frac{n}{n-1}R)} \right]^2 - (nR - (n-2)H^2)^2 \\ & = (n-2)^2 H^4 - \frac{n(n-2)^2}{n-1} H^2 R - n^2 R^2 + 2n(n-2)RH^2 - (n-2)^2 H^4 \\ & = R \left[ -\frac{n(n-2)^2}{n-1} H^2 - n^2 R + 2n(n-2)H^2 \right] \\ & = R \left[ \frac{n^2(n-2)}{n-1} H^2 - n^2 R \right] \\ & < 0. \end{aligned}$$

Hence we have  $\text{Ric}(e_n, e_n) > 0$ . By changing the order of  $\{e_i\}$ , we can show that  $\text{Ric}(e_i, e_i) > 0$  for all  $1 \leq i \leq n$ . So we have  $\lambda_i(H - \lambda_i) > 0$  for all  $i$ . If  $H \geq 0$ , then all  $\{\lambda_i\}$  are positive (if  $\lambda_i < 0$  for some  $i$ , then  $\lambda_i(H - \lambda_i) < 0$ , a contradiction). Similarly, if  $H < 0$ , then all  $\{\lambda_i\}$  are negative. Hence the sectional curvature is positive at  $p$  by (1.3). Since  $M$  is compact and the second fundamental form of  $M$  is definite everywhere, the Gauss map  $M \rightarrow S^n$  is a diffeomorphism [5].  $\square$

*Remark.* Let  $M$  be any hypersurface in  $\mathbf{R}^{n+1}$ . With the same notation as in the preliminaries, we have  $\lambda_1^2 + \dots + \lambda_n^2 \geq H^2/n$  by the Schwartz inequality. Hence

$$R = \sum_{i \neq j} \lambda_i \lambda_j = H^2 - (\lambda_1^2 + \dots + \lambda_n^2) \leq \frac{n-1}{n} H^2.$$

Hence the condition  $R > \frac{n-2}{n-1}H^2$  means actually that  $\frac{n-2}{n-1}H^2 < R \leq \frac{n-1}{n}H^2$ .

**Corollary 2.3.** *Let  $M$  be any hypersurface in  $\mathbf{R}^{n+1}$  and  $R \geq \frac{n-2}{n-1}H^2$ , where  $R$  is the scalar curvature and  $H$  is the mean curvature. Then the sectional curvature is nonnegative.*

*Proof.* It is clear by the proof of the above theorem.  $\square$

Next, we prove a theorem on a complete hypersurface with constant mean curvature.

**Theorem 2.4.** *Let  $M$  be a complete hypersurface in  $\mathbf{R}^{n+1}$  with constant mean curvature  $H$  and  $R \geq \frac{n-2}{n-1}H^2$ , where  $R$  is the scalar curvature. Then  $M$  is a hyperplane or a sphere or a cylinder, that is,  $\mathbf{R}^n$  or  $S^n$  or  $S^{n-1} \times \mathbf{R}^1$ .*

*Proof.* If  $R \geq \frac{n-2}{n-1}H^2$ , then by Corollary 2.3, the sectional curvature is nonnegative. Since  $M$  has constant mean curvature, by Theorem 5 of [3],  $M$  is a generalized cylinder, that is,  $M = S^k \times \mathbf{R}^{n-k}$  for some  $0 \leq k \leq n$ . Hence we may assume that if  $\{\lambda_1, \dots, \lambda_n\}$  are the principal curvatures, then

$$\lambda_i = \begin{cases} \lambda, & \text{for } 1 \leq i \leq k, \\ 0, & \text{for } k+1 \leq i \leq n, \end{cases}$$

everywhere on  $M$ . Hence  $R = \sum_{i \neq j} \lambda_i \lambda_j = k(k-1)\lambda^2 = k(k-1)\left(\frac{H}{k}\right)^2 = \frac{k-1}{k}H^2$ .

So we have  $R = \frac{n-1}{n}H^2$  everywhere or  $R = \frac{n-2}{n-1}H^2$  everywhere on  $M$ . In the first case,  $M = \mathbf{R}^n$  (if  $H = 0$ ) or  $M = S^n$  (if  $H \neq 0$ ) and in the second case,  $M = S^{n-1} \times \mathbf{R}^1$ . □

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