GRÖBNER DUALITY AND MULTIPLE POINTS
IN LINEARLY GENERAL POSITION

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ABSTRACT. It is proved for each d, $1 \leq d \leq n-1$, that a primary 0-dimensional scheme in $\mathbb{P}^n$ of degree $n+2+d$ in linearly general position lies in a rational normal scroll of dimension d.

1. INTRODUCTION

Gröbner, in a series of papers and in his book [4], developed a theory in order to understand both multiplicity for primary ideals at the origin and Macaulay's notion of inverse systems. Gröbner's starting point was just to generalize the obvious univariate case, where α is a root of $f(X)$ of multiplicity d iff $\frac{\partial^n}{\partial X^n}(f)(\alpha) = 0 \ \forall n, 0 \leq n < d$; he proved that if the origin is a root of a 0-dimensional ideal $\mathfrak{I} \subset \mathcal{P} = K[X_1, \ldots, X_n]$ (K any field of characteristic 0), then there are finitely many linear combinations D_i of partial derivatives $\frac{\partial^{i_1 + \ldots + i_n}}{\partial X_1^{i_1} \cdots \partial X_n^{i_n}}$, such that every polynomial $f \in \mathfrak{I}$ satisfies $D_i(f)(0) = 0, \forall i$. Moreover $\{f \in \mathcal{P} : D_i(f)(0) = 0, \forall i\}$ is exactly the primary component at 0 of $\mathfrak{I}$. The D_i's are a basis of a so called σ-stable subspace of differential conditions at 0.

This theory by Gröbner has been systematized and its applications to polynomial system solving have been investigated in [5], [6], where it was christened Gröbner Duality. This paper proposes a different application of Gröbner Duality toward points in linearly general position.

Configurations of projective points in linearly general position (i.e. no $k + 2$ of them are contained in a plane of dimension k) have been much investigated since such configurations arise by cutting a variety of dimension d with a linear variety of dimension $n - d$. The old Castelnuovo result that such a configuration of $n + 3$ points in $\mathbb{P}^n$ lies in a rational normal curve has been generalized in the context of schemes [3]; around the Castelnuovo Theory, there arose the question whether a suitable configuration lies in a rational normal scroll. In [2] it was proved for each d, $1 \leq d \leq n - 2$, that a configuration of $n + 2 + d$ points in $\mathbb{P}^n$ lies in a rational normal scroll of dimension d.

Let $\mathfrak{q}$ be a primary ideal at the origin in linearly general position and dimension $\mu \geq n + 3$, so – by the result in [3] – of embedding dimension 1; let us denote by $\mathfrak{q}_0$, $\mathfrak{q}_1$ the unique primary ideals s.t. $\mathfrak{q}_i \supset \mathfrak{q}$, $\mathrm{mult}(\mathfrak{q}_0) = n + 1$, $\mathrm{mult}(\mathfrak{q}_1) = n + 3$. The

approach which allowed me to generalize the result of [2] to the case of primary ideals at the origin, is quite straightforward:

- Gröbner Duality allowed me to fix a suitable frame of coordinates $Y_1, \ldots, Y_n$ s.t. $\mathfrak{q}_0$ contains the ideal of the rational normal curve with parametric equations $Y_i = t^i, 1 \le i \le n$ (Prop. 11).
- This allowed me to describe the differential conditions defining $\mathfrak{q}_0$ (Lemma 9)
- and (by continuation) those defining $\mathfrak{q}_1$ and $\mathfrak{q}$ (Prop. 12);
- the basic duality of Gröbner theory then allows me to compute the ideal $\mathfrak{I}$ defining the unique rational normal curve s.t. $\mathfrak{I} \subset \mathfrak{q}_1$ (Prop. 16), whose existence and uniqueness are proved in [3].
- A further application of Gröbner Duality allowed me to verify that $\mathfrak{q}$ lies in a rational normal scroll of the expected dimension.

2. Preliminaries

2.1. Gröbner Duality. Let $\mathcal{P} = K[Y_1, \ldots, Y_n]$ and let $\mathbf{T}$ be the semigroup generated by $\{Y_1, \ldots, Y_n\}$.

$D(i_1, \ldots, i_n) : \mathcal{P} \to \mathcal{P}$ defines the differential operator

$$D(i_1, \ldots, i_n) = \frac{1}{i_1! \ldots i_n!} \frac{\partial^{i_1 + \cdots + i_n}}{\partial Y_1^{i_1} \cdots \partial Y_n^{i_n}}.$$

This notation will however be simplified by writing $D(t) := D(i_1, \ldots, i_n)$ where $t = Y_1^{i_1} \ldots Y_n^{i_n}$; we moreover denote

$$\mathbf{D} := \{D(t) : t \in \mathbf{T}\}, \quad \mathbf{D}^{(i)} := \{D(t) \in \mathbf{D} : \deg(t) \le i\}$$

and $\mathrm{Span}_K(\mathbf{D})$ denotes the K-vector space generated by $\mathbf{D}$.

If $X_1, \ldots, X_j \in \mathcal{P}$ are linearly independent linear forms, the definition of $D(t)$ is obviously generalized to define $D(\tau) \in \mathrm{Span}_K(\mathbf{D})$, for each $\tau \in \langle X_1, \ldots, X_j \rangle$; on the basis of that we generalize the notation above to denote

$$\mathbf{D}(X_1, \ldots, X_j) := \{D(t) : t \in \langle X_1, \ldots, X_j \rangle\},$$

$$\mathbf{D}^{(i)}(X_1, \ldots, X_j) := \{D(t) \in \mathbf{D}(X_1, \ldots, X_j) : \deg(t) \le i\}.$$

For each $j = 1 \ldots n$, $\sigma_j = \sigma_{Y_j} : \mathrm{Span}_K(\mathbf{D}) \to \mathrm{Span}_K(\mathbf{D})$ is the antiderivative with respect to Y_j, i.e. the linear map s.t.:

$$\sigma_{Y_j}(D(i_1, \ldots, i_n)) = \begin{cases} D(i_1, \ldots, i_j - 1, \ldots, i_n) & \text{if } i_j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sigma_{Y_j} \sigma_{Y_i} = \sigma_{Y_i} \sigma_{Y_j} \forall i, j$, the linear maps σ_t are obviously defined $\forall t \in \mathbf{T}$.
A K-vector subspace $V \subset \mathrm{Span}_K(\mathbf{D})$ is σ-*stable* if $\forall t \in \mathbf{T}$, $\forall L \in V$, $\sigma_t(L) \in V$.
If V is a σ-stable subspace of $\mathrm{Span}_K(\mathbf{D})$, let

$$\Im(V) := \{f \in \mathcal{P} : L(f)(0) = 0 \ \forall L \in V\}.$$

Let $\mathfrak{m} = (Y_1, \ldots, Y_n) \subset \mathcal{P}$; if $\mathfrak{I} \subset \mathfrak{m}$ is an ideal, define

$$\Delta(\mathfrak{I}) := \{L \in \mathrm{Span}_K(\mathbf{D}) : L(f)(0) = 0 \ \forall f \in \mathfrak{I}\}.$$

Theorem 1. *There is a one-to-one correspondence between $\mathfrak{m}$-primary ideals of $\mathcal{P}$ and finite dimensional σ-stable subspaces of $\mathrm{Span}_K(\mathbf{D})$. More exactly, every $\mathfrak{m}$-primary ideal $\mathfrak{q}$ corresponds to the finite dimensional σ-stable subspace $\Delta(\mathfrak{q})$, and*

every finite dimensional σ-stable subspace $V \subset \mathrm{Span}_K(\mathbf{D})$ corresponds to the $\mathfrak{m}$-primary ideal $\Im(V)$, so that $\mathfrak{q} = \Im(\Delta(\mathfrak{q}))$ and $V = \Delta(\Im(V))$.

Moreover $\dim_K(\Delta(\mathfrak{q})) = \mathrm{mult}(\mathfrak{q}), \mathrm{mult}(\Im(V)) = \dim_K(V)$.

Proof. Cf. [4], pp. 174–178. $\qquad\square$

Gröbner Duality is generalizable as:

Theorem 2. *$\Im$ and Δ give a one-to-one correspondence between ideals of $\mathcal{P}$ closed w.r.t. the $\mathfrak{m}$-adic topology and σ-stable subspaces of $\mathrm{Span}_K(\mathbf{D})$.*

Let us consider now the linear map $\rho_{Y_j} : \mathrm{Span}_K(\mathbf{D}) \to \mathrm{Span}_K(\mathbf{D})$, $j = 1, \dots, n$, s.t. $\rho_{Y_j}(D(i_1, \dots, i_n)) = D(i_1, \dots, i_j + 1, \dots, i_n)$. Again $\forall t \in \mathbf{T}, \rho_t$ is defined in the obvious way and $\sigma_t \rho_t = \mathrm{Id}$ holds $\forall t \in \mathbf{T}$.

Each $L \in \mathrm{Span}_K(\mathbf{D} \setminus \{Id\})$ can be uniquely written as $L = L_1 + \cdots + L_n$ where

$$L_j \in \mathrm{Span}_K\left(\mathbf{D}(Y_j, \dots, Y_n) \setminus \mathbf{D}(Y_{j+1}, \dots, Y_n)\right).$$

Denote $L^{\geq j} := \sum_{i=j}^n L_i$; analogously, we use also the notation $L^{\leq j}$, $L^{>j}$, $L^{<j}$.

A semigroup ordering $<$ on $\mathbf{T}$ obviously induces an ordering also on $\mathbf{D}$; with respect to it, we define $T(L) \in \mathbf{D}$ for any $L \in \mathrm{Span}_K(\mathbf{D})$ as follows: if $L = \sum c_i D_i$ with $c_i \neq 0$, $D_i \in \mathbf{D}$, $D_1 > D_2 > \cdots$, then $T(L) = D_1$.

Let $U \subset \mathrm{Span}_K(\mathbf{D})$ be σ-stable. We then denote $T(U) = \{T(L) : L \in U\}$, which is σ-stable itself, and

$$\mathcal{C}(U) := \{D(\tau) \in \mathbf{D} : D(\tau) \notin T(U), \sigma_i(D(\tau)) \in T(U) \ \forall i\}.$$

Definition 3. Let $D(t) \in \mathcal{C}(U)$; an element $L \in \mathrm{Span}_K(\mathbf{D})$ s.t.

(c1) $T(L) = D(t)$,

(c2) $\forall j, \sigma_j(L) \in U$,

(c3) if $L = D(t) + \sum c_i D(\tau_i)$ with $c_i \neq 0$, then $\forall i D(\tau_i) \notin T(U)$

will be called a continuation of U at t.

Proposition 4. (1) *If a continuation of U at t exists, then there is a unique one, $C_{U,t}$, satisfying*

(c4) *if $C_{U,t} = D(t) + \sum c_i D(\tau_i)$ with $c_i \neq 0$, then for each $D(\tau_i) \in \mathcal{C}(U)$, there is no continuation of U at τ_i.*

(2) *The following conditions are equivalent:*

a) *$V := U + \mathrm{Span}_K(\{L\})$ is σ-stable.*

b) *There are $t_0 > \cdots > t_s$, $D(t_i) \in \mathcal{C}(U)$ such that $L = C_{U,t_0} + \sum_{i=1}^s c_i C_{U,t_i}$.*

Proof. Cf. [5], Cor. 4.1 and Prop. 4.1. $\qquad\square$

Proposition 5. *Let $\Phi : \mathbf{T} \mapsto \mathbb{Z}^r$ be a semigroup morphism; let $\mathcal{I}_\Phi$ the ideal*

$$\{m_1 - m_2 : m_i \in \mathbf{T}, \Phi(m_1) = \Phi(m_2)\} \subset \mathcal{P}$$

and $\forall \alpha \in \mathbb{Z}^r$, let $\delta_\alpha = \sum_{\substack{t \in \mathbf{T} \\ \Phi(t) = \alpha}} D(t) \in \mathrm{Span}_K(\mathbf{D})$. Then $\Delta(\mathcal{I}_\Phi) = \{\delta_\alpha : \alpha \in \mathbb{Z}^r\}$.

Proof. Let us fix any $\alpha \in \mathbb{Z}^r$ and let $\mathcal{P}_\alpha = \mathrm{Span}_K(\{m \in \mathbf{T} : \Phi(m) = \alpha\}), \mathcal{I}_\alpha = \mathcal{I}_\Phi \cap \mathcal{P}_\alpha$. Obviously $\forall f \in \mathcal{I}_\alpha$ one has $\delta_\alpha(f) = 0$. Also $\forall t \in \mathbf{T}$ s.t. $\Phi(t) = \alpha$ and $\forall m \in \mathbf{T}$ s.t. $\Phi(m) \neq \alpha$ we have $D(t)(m)(0) = 0$, so that $\delta_\alpha(f) = 0, \forall f \in \mathcal{I}_\Phi$. Since moreover $\dim_K(\mathcal{I}_\alpha) = \dim_K(\mathcal{P}_\alpha) - 1$, the claim follows by a vector space dimensional count. $\qquad\square$

2.2. **0-dim. primary ideals in l.g.p. and rational normal scrolls.**

Definition 6. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. $\mathfrak{q}$ is said to be in linearly general position (l.g.p.) if for each linear primary ideal $\wp \subset \mathfrak{m}$ we have $\mathrm{mult}(\mathfrak{q} + \wp) \leq 1 + \dim(\wp)$.

Definition 7. A rational normal scroll is the variety defined by the ideal $\mathcal{R}(M) \subset \mathcal{P}$ generated by the minors of a matrix

$$M = \begin{pmatrix} \ell_{11} & \dots & \ell_{1j} & \dots & \ell_{1k} \\ \ell_{21} & \dots & \ell_{2j} & \dots & \ell_{2k} \end{pmatrix}$$

such that

- $\ell_{ij} \in \mathrm{Span}_K(\{1, Y_1, \dots, Y_n\}), \forall 1 \leq i \leq 2, 1 \leq j \leq k$;
- $\lambda\ell_{11} + \mu\ell_{21}, \dots \lambda\ell_{1j} + \mu\ell_{2j}, \dots \lambda\ell_{1k} + \mu\ell_{2k}$ are linearly independent $\forall (\lambda, \mu) \in \mathbb{P}^2$.

In the case of a rational normal curve $(k = n)$, I need the following fact.

Lemma 8. *Let M be a matrix as above $(k = n)$ and such that, moreover,*

(1) $\mathcal{R}(M) \subset \mathfrak{m}$,
(2) $\mathcal{R}(M) \not\subset \mathfrak{m}^2$,
(3) $D(Y_1) \in \Delta(\mathcal{R}(M))$.

Then there are $c_k \in K, 1 \leq k \leq n$, and linear forms $\mathbf{l}_i, 2 \leq i \leq n$, s.t., denoting

$$(1) \qquad \mathcal{M} = \begin{pmatrix} Y_1 & Y_2 & \dots & Y_j & \dots & Y_n \\ 1 + \sum_{k=1}^{n} c_k Y_k & \mathbf{l}_2 & \dots & \mathbf{l}_j & \dots & \mathbf{l}_n \end{pmatrix},$$

one has $\mathcal{R}(M) = \mathcal{R}(\mathcal{M})$.

Proof. By 1 and 2, up to permutations and operations of rows and columns we can assume that each form ℓ_{ij} except ℓ_{21} is s.t. $\ell_{ij}(0) = 0$, and $\ell_{21}(0) = 1$, so that

$$\ell_{21} = 1 + \sum_{k=1}^{n} \ell_{21}^{(k)} Y_k, \qquad \ell_{ij} = \sum_{k=1}^{n} \ell_{ij}^{(k)} Y_k \quad \forall(i,j) \neq (2,1).$$

Let $M^{(\lambda\rho)} = \ell_{1\lambda}\ell_{2\rho} - \ell_{2\lambda}\ell_{1\rho}$ be the minor over the λth and ρ^{th} columns $(\lambda < \rho)$. By 3 we have

$$0 = D(Y_1)(M^{(1\rho)})(0) = \ell_{11}^{(1)}\ell_{2\rho}(0) + \ell_{11}(0)\ell_{2\rho}^{(1)} - \ell_{21}^{(1)}\ell_{1\rho}(0) - \ell_{21}(0)\ell_{1\rho}^{(1)} = -\ell_{1\rho}^{(1)},$$

which implies $\ell_{1\rho}^{(1)} = 0$ for $\rho > 1$. Since $\{\ell_{1\rho}, \rho = 2, \dots, n\}$ are linearly independent, column operations can be used to modify the first column so that $\ell_{11} = Y_1$. For the same reason, further column operations not affecting the first column can now allow us to assume that $\ell_{1j} = Y_j$. $\qquad\square$

3. RELATIONS

Let us impose a second graduation wt on $\mathcal{P}$ by setting $wt(Y_i) = i$, and let us denote $\mathcal{R} = (Y_2 - Y_1^2, \dots, Y_n - Y_1^n)$ and $\forall j$: $\mathbf{D}(j) = \{t \in \mathbf{D}(Y_1, \dots, Y_{j-1}) : wt(t) = j\}, \delta_j = \sum_{\substack{t \in \mathbf{T} \\ wt(t)=j}} D(t), \delta'_j = \sum_{t \in \mathbf{D}(j)} D(t), U_j = \mathrm{Span}_K(\delta_0, \dots, \delta_j)$. We also fix as semigroup ordering $<$ on $\mathbf{T}$ the lexicographical ordering s.t. $Y_1 > Y_2 > \dots > Y_n$.

Lemma 9. *The following hold:*

(1) $\Delta(\mathcal{R}) = \{\delta_j : 0 \leq j\}$.

(2) $\sigma_i(\delta_j) = \delta_{j-i}, \ \forall i,j : i \leq j$.

(3) $\forall j, T(\delta_j) = Y_1^j$.

(4) $\forall j, \mathcal{C}(U_j) = \{D(Y_1^{j+1}), D(Y_2), \ldots, D(Y_n)\}$.

(5) The generic continuations of U_{j-1} are $\delta_j' + \sum_{k=2}^n c_k D(Y_k)$.

(6) $\forall j, \delta_j = \sum_{i=1}^j \rho_i(\delta_{j-i}^{\geq i})$.

Proof. 1 is a corollary of Proposition 5; 2, 3 and 6 require just an obvious verification; 3 implies $T(U_j) = \{1, Y_1, \ldots, Y_1^j\}$ which is equivalent to 4.

Ad 5: First of all remark that δ_j' is a continuation: c1) and c3) are obvious and c2) follows from 2. Then 5 follows from this, Prop. 4 and the obvious fact that $D(Y_k), k > 1$, is a continuation. $\qquad\square$

Corollary 10. *Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of multiplicity $\mu = j + 1$, $j \leq n$, and embedding dimension 1 and such that $\Delta(\mathfrak{q}) \cap \mathrm{Span}_K(\mathbf{D}^{(1)}) = \{Id, D(Y_1)\}$. If $U_{j-1} \subset \Delta(\mathfrak{q})$, then $\Delta(\mathfrak{q}) = U_{j-1} + \mathrm{Span}_K(\{\delta\})$, where $\delta = \delta_j' + \sum_{k=2}^n c_k D(Y_k)$ for some c_k.*

Proof. Since $\dim_K(\Delta(\mathfrak{q})) = j + 1 = \dim_K(U_{j-1}) + 1$ and $U_{j-1} \subset \Delta(\mathfrak{q})$, it follows that $\Delta(\mathfrak{q}) = U_{j-1} + \mathrm{Span}_K(\{\delta\})$; then δ is a continuation of U_{j-1}. If $T(\delta) = D(Y_l)$ we have $\Delta(\mathfrak{q}) \cap \mathrm{Span}_K(\mathbf{D}^{(1)}) = \{Id, D(Y_1), \delta\}$, which implies that the embedding dimension of $\mathfrak{q}$ is 2 (cf. [1], prop. 2) and gives a contradiction. So $T(\delta) = D(Y_1^\mu)$. The claim then follows from this, Prop. 4 and Lemma 9.5. $\qquad\square$

Proposition 11. *Let $\mathfrak{q} \subset K[X_1, \ldots, X_n]$ be an $\mathfrak{m}$-primary ideal of multiplicity μ and s.t. its embedding dimension is 1.*

The following conditions are then equivalent:

(1) $\mathfrak{q}$ *is in l.g.p.;*

(2) *there is a coordinate frame $Y_1, \ldots, Y_n$ s.t.*

$$\mathfrak{q} = \begin{cases} (Y_1^\mu, Y_2 - Y_1^2 - Y_1^{n+1}f_2(Y_1), \ldots, Y_n - Y_1^n - Y_1^{n+1}f_n(Y_1)) & \text{if } \mu > n, \\ (Y_1^\mu, Y_2 - Y_1^2, \ldots, Y_{\mu-1} - Y_1^{\mu-1}, Y_\mu, \ldots, Y_n) & \text{if } \mu \leq n; \end{cases}$$

(3) *there is a coordinate frame $Y_1, \ldots, Y_n$ s.t., setting $\nu = min(n, \mu - 1)$:*

    - $\forall i, 1 \leq i \leq \nu, V_i := \Delta(\mathfrak{q}) \cap \mathrm{Span}_K\big(\mathbf{D}^{(i)}(Y_1, \ldots, Y_i)\big) = U_i$;

    - $D(Y_i)(Y_j) = 0$ *iff* $1 \leq i \leq \nu < j \leq n$.

Proof. $1 \Longrightarrow 3$. Since $\mathfrak{q}$ has embedding dimension 1, then ([1], prop. 2) there is Y_1 s.t. V_1 is generated by $\{Id, D(Y_1)\}$. Now let us assume that for some $i < \nu$ we have found linearly independent forms $Y_1, \ldots, Y_i$ s.t. $V_i = U_i$ (this holds at least for $i = 1$, giving the basis for the induction) and let us fix linear forms $Z_{i+1}, \ldots, Z_n$ s.t. $\mathfrak{m} = (Y_1, \ldots, Y_i, Z_{i+1}, \ldots, Z_n)$. Our aim is to prove that there is a further linear form Y_{i+1} linearly independent with $Y_1, \ldots, Y_i$ s.t. $V_{i+1} = U_{i+1}$, which is sufficient to prove the statement by the inductive argument, since the vector space of the linear forms Z s.t. $D(Y_i)(Z) = 0 \ \forall i \leq \nu$ is orthogonal to $\mathrm{Span}_K(Y_1, \ldots, Y_\nu)$.

To prove that, notice that, as $\dim_K(\Delta(\mathfrak{q})) = \mathrm{mult}(\mathfrak{q}) = \mu > i + 1 = \dim(U_i)$, there is $\delta \in \Delta(\mathfrak{q}) \setminus U_i$ s.t. δ is a continuation of U_i; then by Corollary 10

$$\delta = \delta_{i+1}' + \sum_{k=2}^i c_k D(Y_k) + \sum_{k=i+1}^n c_k D(Z_k) \quad \text{for some} \quad c_k.$$

Let us denote $Y_{i+1} := \sum_{k=2}^{i} c_k D(Y_k) + \sum_{k=i+1}^{n} c_k D(Z_k)$. To prove that $Y_{i+1} \notin \mathrm{Span}_K(\{Y_1, \dots, Y_i\})$, let us consider $\wp = (Z_{i+1}, \dots, Z_n)$; then $\Delta(\mathfrak{q} + \wp) = \Delta(\mathfrak{q}) \cap \Delta(\wp) = \Delta(\mathfrak{q}) \cap \mathrm{Span}_K(\mathbf{D}(Y_1, \dots, Y_i)) = V_i$.

Therefore if $Y_{i+1} \in \mathrm{Span}_K(\{Y_1, \dots, Y_i\})$, then $\{\delta_0, \dots, \delta_i, \delta\} \subset \Delta(\mathfrak{q} + \wp)$, so that $\mathrm{mult}(\mathfrak{q} + \wp) = \dim_K(\Delta(\mathfrak{q} + \wp)) > i + 1$, against the assumption that $\mathfrak{q}$ is in l.g.p.

We then have $\delta = D(Y_{i+1}) + \delta'_{i+1} = \delta_{i+1}$, so that $V_{i+1} \supseteq U_{i+1}$. Moreover if $V_{i+1} \neq U_{i+1}$ then it would contain a continuation L of U_{i+1}; according to Lemma 9, $T(L) = D(Y_1^{i+2})$, so $L \notin \mathbf{D}^{(i+1)}$. By this contradiction, we conclude that $V_{i+1} = U_{i+1}$.

$3 \iff 2$. Since $\mathrm{Span}_K(\mathbf{D}^{(i)}) = \Delta(\mathfrak{m}^{i+1})$, Lemma 9.1 implies that for $i \leq n$

$$U_i = \mathrm{Span}_K(\delta_0, \dots, \delta_i) = \mathrm{Span}_K(\{\delta_j, j \geq 0\}) \cap \mathrm{Span}_K(\mathbf{D}^{(i)}) = \Delta(\mathcal{R} + \mathfrak{m}^{i+1}),$$

so that

$$\mathrm{Span}_K(\delta_0, \dots, \delta_i) \subset \Delta(\mathfrak{q})$$

$$\iff \mathfrak{q} \subset \mathcal{R} + \mathfrak{m}^{i+1} = \begin{cases} (Y_1^{n+1}, Y_2 - Y_1^2, \dots, Y_n - Y_1^n) & \text{if } i = n, \\ (Y_1^{i+1}, Y_2 - Y_1^2, \dots, Y_i - Y_1^i, Y_{i+1}, \dots, Y_n) & \text{if } i < n; \end{cases}$$

which is of course equivalent to 2, since $\mathfrak{q}$ has embedding dimension 1.

$2 \implies 1$. Let $\mathfrak{q} = (Y_1^\mu, Y_2 - Y_1^2 - Y_1^{n+1} f_2(Y_1), \dots, Y_n - Y_1^n - Y_1^{n+1} f_n(Y_1))$ and let $\wp$ be a linear ideal thru the origin. Let $k < n$ be the maximal index s.t. $\wp \subset (Y_{k+1}, \dots, Y_n)$. Then

$$\mathfrak{q} + \wp \subset (Y_1^\mu, Y_2 - Y_1^2 - Y_1^{n+1} f_2(Y_1), \dots, Y_n - Y_1^n - Y_1^{n+1} f_n(Y_1), Y_{k+1}, \dots, Y_n)$$

$$= (Y_1^\mu, Y_1^{k+1}, Y_2 - Y_1^2, \dots, Y_k - Y_1^k, Y_{k+1}, \dots, Y_n),$$

so its multiplicity is at most $k + 1$ while the dimension of $\wp$ is at least k. □

4. Differential conditions of l.g.p. $\mathfrak{m}$-primary ideals

As a consequence of Prop. 11, if $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of multiplicity $\mu > n+1$, of embedding dimension 1 and in l.g.p., we can assume that it is contained in $\mathfrak{q}_0 = \Im(\mathrm{Span}_K(\{\delta_0, \dots, \delta_n\}))$ and that a basis of $\mathfrak{q}$ is $\mathcal{B} = (Y_1^\mu, f_2, \dots, f_n)$, where

$$f_j = Y_j - Y_1^j - \sum_{i=1}^{\mu-n-1} d_{ji} Y_1^{n+i}.$$

Our aim is now to describe $\Delta(\mathfrak{q})$ in terms of d_{ji}.

Proposition 12. $\Delta(\mathfrak{q}) = \{L_0, \dots, L_n, L_{n+1}, \dots, L_{\mu-1}\}$, where

$$L_i = \delta_i \; \forall i \leq n,$$

$$L_{n+1} = \delta_{n+1} + \sum_{j=2}^{n} d_{j1} D(Y_j),$$

$$L_{n+2} = \delta_{n+2} + \sum_{j=2}^{n} d_{j1} D(Y_1 Y_j) + \sum_{j=2}^{n} d_{j2} D(Y_j),$$

and in general

$$L_{n+k} = \sum_{j=1}^{n} \rho_j(L_{n+k-j}^{\geq j}) + \sum_{j=2}^{n}\sum_{i=1}^{k} d_{ji}\rho_j(L_{k-i}^{\geq j}).$$

Proof. One has to prove that:

(1) $L_{n+k}(f)(0) = 0$ for each $f \in \mathcal{B}$;

(2) $\sigma_j(L_{n+k}) \in \mathrm{Span}_K(\Delta(\mathfrak{q}))$ for each j.

Ad 1: remark that $L_{n+k} = D(Y_1^{n+k}) + \tilde{L}_{n+k} + \sum_{j=2}^{n} d_{jk}D(Y_j)$, where $\tilde{L}_{n+k}$ is a combination of terms $D(Y_i t)$ s.t. $i > 1, t \neq 1$, so that $\tilde{L}_{n+k}(f)(0) = 0 \ \forall f \in \mathcal{B}$. It is immediate to verify that $D(Y_1^{n+k})(f_i)(0) = -d_{ik} = -\sum_{j=2}^{n} d_{jk}D(Y_j)(f_i)(0)$.

Ad 2: A direct computation allows us to verify that:

- $\sigma_1(L_{n+k}) = L_{n+k-1}$;
- $\sigma_l(L_{n+k}) = L_{n+k-l} + \sum_{i=1}^{k} d_{li}L_{k-i}, l > 1$.

$\square$

5. Normal rational curves containing l.g.p. $\mathfrak{m}$-primary ideals

This section will deal with the following

Problem. Given an l.g.p. $\mathfrak{m}$-primary ideal $\mathfrak{q}_1 \subset \mathcal{P}$ s.t. $\mathrm{mult}(\mathfrak{q}_1) = \mu = n + 3$, explicitly construct the unique rational normal curve $\mathcal{I}$ s.t. $\mathfrak{q}_1 \supset \mathcal{I}$.

The existence of $\mathcal{I}$ is assured by [3], Th. 1. Of course such a construction does not give at all a proof of [3], Th. 1; in fact, our construction is based on the fact that we can fix a frame of coordinates satisfying conditions 2 and 3 of Prop. 11; it is [3], Th. 1 itself which guarantees the existence of such a frame since it lets us apply Prop. 11 guaranteeing that $\mathfrak{q}_1$ has embedding dimension 1.

As a consequence of Lemma 8, we know that if $\mathcal{I}$ is a rational normal curve there is a matrix $\mathcal{M}$ as in (1) s.t. $\mathcal{R}(\mathcal{M}) = \mathcal{I}$. In the rest of the argument we will write $\mathbf{l}_j = \sum_{k=1}^{n} \mathbf{l}_j^{(k)} Y_k$ for $j \geq 2$, we will set $\mathbf{l}_1 = 1 + \sum_{k=1}^{n} c_k Y_k$ and we will denote by $\mathcal{M}_{\mu\nu}, \forall \mu, \nu$ the matrices

$$\mathcal{M}_{\mu\nu} = \begin{pmatrix} D(Y_\mu)(Y_1) & D(Y_\mu)(Y_2) & \dots & D(Y_\mu)(Y_n) \\ D(Y_\nu)(\mathbf{l}_1) & D(Y_\nu)(\mathbf{l}_2) & \dots & D(Y_\nu)(\mathbf{l}_n) \end{pmatrix}.$$

For a $2 \times n$ matrix M we will again write $M^{(\lambda\rho)}$ to denote the minor over the λth and ρ^{th} columns.

As a preliminary step, let us consider an l.g.p. $\mathfrak{m}$-primary ideal $\mathfrak{q}_0$ s.t. $\mathrm{mult}(\mathfrak{q}_0) = n + 1$ and try to compute all the rational normal curves passing thru it. We first fix a frame of coordinates $Y_1, \dots, Y_n$ such that conditions 2 and 3 are satisfied by $\mathfrak{q}_0$.

Remark 13. (1) $D(Y_\mu Y_\nu)(\mathcal{M}^{(\lambda\rho)})(0) = \mathcal{M}_{\mu\nu}^{(\lambda\rho)} + \mathcal{M}_{\nu\mu}^{(\lambda\rho)}$.

(2) $\mathcal{R}(\mathcal{M}) \subset \mathfrak{q}_0 \iff \delta_j(\mathcal{M}^{(\lambda\rho)})(0) = 0 \ \forall \lambda, \rho, j = 1 \dots, n$
$\iff \delta_j^*(\mathcal{M}^{(\lambda\rho)})(0) = 0 \ \forall \lambda, \rho, j = 1 \dots, n$

where δ_j^* is the projection of δ_j over $\mathrm{Span}_K(\mathbf{D}^{(2)})$, i.e.

$$\delta_i^* = D(Y_i) + \sum_{j \leq i/2} D(Y_j Y_{i-j}).$$

(3) $\delta_1^*(\mathcal{M}^{(\lambda\rho)})(0) = 0 \ \forall \lambda, \rho.$

(4) $0 = \delta_2^*(\mathcal{M}^{(\lambda\rho)})(0) = \begin{cases} \mathbf{l}_2^{(1)} - 1 & \text{if } (\lambda, \rho) = (1, 2), \\ \mathbf{l}_j^{(1)} & \text{if } (\lambda, \rho) = (1, j), \ 2 < j, \\ 0 & \text{otherwise.} \end{cases}$

(5) $D(Y_{i+1})(\mathcal{M}^{(\lambda\rho)})(0) = \begin{cases} -1 & \text{if } (\lambda, \rho) = (1, i+1), \\ 0 & \text{otherwise.} \end{cases}$

Proposition 14. *The following conditions are equivalent:*

(1) $\mathcal{R}(\mathcal{M}) \subset \mathfrak{q}_0$;

(2) $\forall j, 2 \leq j \leq n$ there is d_j s.t. $\mathbf{l}_j = Y_{j-1} + \sum_{k=j}^{n-1} c_{k-j+1} Y_k + d_j Y_n$.

Proof. Clearly $2 \implies 1$ is immediate, so let us focus on the proof of $1 \implies 2$. The argument goes by induction on i: one assumes that:

$$(\mathbf{A}(i)) \qquad\qquad \forall k < i \qquad \mathbf{l}_j^{(k)} = \begin{cases} 0 & \text{if } k < j - 1, \\ 1 & \text{if } k = j - 1, \\ c_{k-j+1} & \text{if } j \leq k < i, \end{cases}$$

and proves the case $\mathbf{A}(i+1)$; note that $\mathbf{A}(2)$ is guaranteed by Rem. 13.4. As the case $\mathbf{A}(2)$ was derived by computing $\delta_2^*(\mathcal{M}^{(\lambda\rho)})(0)$, we will deduce the case $\mathbf{A}(i+1)$ computing $\delta_{i+1}^*(\mathcal{M}^{(\lambda\rho)})(0)$. Under the inductive assumption $\mathbf{A}(i)$ one has

$$0 = \delta_{i+1}^*(\mathcal{M}^{(\lambda\rho)})(0) = \begin{cases} \mathbf{l}_\rho^{(i)} - c_{i+1-\rho} & \text{if } \lambda = 1, \rho \leq i, \\ \mathbf{l}_{i+1}^{(i)} - 1 & \text{if } \lambda = 1, \rho = i+1, \\ \mathbf{l}_\rho^{(i)} & \text{if } \lambda = 1, \rho > i+1, \\ 0 & \text{otherwise,} \end{cases}$$

whence we obtain $\mathbf{A}(i+1)$ and so the claim. $\qquad\qquad\qquad\qquad \square$

Let us now assume we have an l.g.p. $\mathfrak{m}$-primary ideal $\mathfrak{q}_1$ s.t. $\mathrm{mult}(\mathfrak{q}_1) = n + 3$, and let us try to compute the rational normal curve passing thru it. Since $\mathfrak{q}_1$ has embedding dimension 1, we know it is contained in a unique $\mathfrak{m}$-primary ideal $\mathfrak{q}_0 \supset \mathfrak{q}_1$ s.t. $\mathrm{mult}(\mathfrak{q}_0) = n + 1$.

The knowledge of $\mathfrak{q}_0$ allows us to fix a frame $Y_1, \ldots, Y_n$ and to compute a matrix $\mathcal{M}$ s.t. the rational normal curve $\mathcal{R}(\mathcal{M})$ contains $\mathfrak{q}_0$ (Prop. 14).

Remark 15. A quadratic polynomial f is in $\mathfrak{q}$ if and only if $L^*(f)(0) = 0 \ \forall L \in \Delta(\mathfrak{q})$, where L^* denotes the projection of L over $\mathrm{Span}_K(\mathbf{D}^{(2)})$.

Proposition 16. *In the above notation the only rational normal curve containing $\mathfrak{q}_1$ is $\mathcal{R}(\mathbf{M})$, where $\mathbf{M}$ is the matrix*

$$\begin{pmatrix} Y_1 & \ldots & Y_j & \ldots & Y_n \\ Y_0 + \sum_{i=2}^n c_i Y_i & \ldots & Y_{j-1} + \sum_{i=j+1}^n c_{i+1-j} Y_i + d_{j1} Y_n & \ldots & Y_{n-1} + d_{n1} Y_n \end{pmatrix}$$

and c_i, $2 \leq i \leq n$, are functions of d_{j1} and d_{j2} given by

$$(**) \qquad \begin{cases} \sum_{j=3}^n d_{j1} c_{j-1} + d_{n1} d_{21} - c_n - d_{22} = 0, \\ d_{\rho-1\,1} + \sum_{j=\rho+1}^n d_{j1} c_{j+1-\rho} + d_{n1} d_{\rho1} - c_{n+2-\rho} - d_{\rho2} = 0, \quad \rho > 2. \end{cases}$$

Proof. To prove the statement we have to compute which conditions are imposed on the minors $\mathcal{M}^{(\lambda\rho)}$ by L_{n+1}, L_{n+2}, i.e. we have just to solve the equations

$$L_i^*(\mathcal{M}^{(\lambda\rho)})(0) = 0, \quad i = n+1, n+2, 1 \le \lambda < \rho \le n,$$

where

$$L_{n+1}^* = \sum_{j=1}^{n/2} D(Y_j Y_{n+1-j}) + \sum_{i=2}^{n} d_{i1} D(Y_i),$$

$$L_{n+2}^* = \sum_{j=2}^{n/2} D(Y_j Y_{n+2-j}) + \sum_{i=2}^{n} d_{i1} D(Y_1 Y_i) + \sum_{i=2}^{n} d_{i2} D(Y_i).$$

A direct computation shows that

$$L_{n+1}^*(\mathcal{M}^{(\lambda\rho)})(0) = 0, \forall \lambda, \rho \implies \mathbf{l}_\rho^{(n)} = c_{n-\rho+1} + d_{\rho 1}, \forall \rho \ge 2.$$

After substituting these values in $\mathcal{M}$, note that the coefficient of Y_i in $\mathcal{M}_{2i}$ is c_1, so we can subtract the first row from the second c_1 times and obtain the matrix $\mathbf{M}$. Moreover $L_{n+2}^*(\mathbf{M}^{(\lambda\rho)})(0) = 0, \forall \lambda, \rho$ gives (**). $\qquad \square$

6. Normal rational scrolls containing l.g.p. $\mathfrak{m}$-primary ideals

It is quite natural to continue the same computation we did in the previous paragraph, basing ourself on Proposition 12, to study where an l.g.p. $\mathfrak{m}$-primary ideal $\mathfrak{q}$ with $\text{mult}(\mathfrak{q}) = \mu > n + 3$ lies.

We already know that $\Delta(\mathfrak{q}) = \{L_0, \dots, L_n, L_{n+1}, \dots, L_{\mu-1}\}$ and that for $\mathbf{M}$ defined as in Prop. 16 we have $\mathcal{R}(\mathbf{M}) \subset \Im(\text{Span}_K(\{L_0, \dots, L_{n+2}\})) = \mathfrak{q}_1$, so we have just to compute $L_j(\mathbf{M}^{(\lambda\rho)})$ to derive our conclusions. To follow this plan we need to introduce some notation and fix some details.

Remark 17. For each $\lambda > 1$, $\mathbf{M}^{(\lambda\rho)}$ does not depend on $Y_0, \dots, Y_{\lambda-2}$ nor on the terms $Y_{\lambda-1}^2, Y_{\lambda-1}Y_\lambda$ nor on linear terms Y_j. In order not to write operators which are clearly zero, let us therefore denote by $\mathbf{D}^{[\lambda]}$ the set of elements

$$\mathbf{D}^{[\lambda]} = \{D(Y_i Y_j) : i \le j, i \ge \lambda\} \cup \{D(Y_{\lambda-1}Y_j : j > \lambda\}$$

and for $L \in \text{Span}_K(\mathbf{D})$ let us denote by $L^{[\lambda]}$ its projection over $\text{Span}_K(\mathbf{D}^{[\lambda]})$; then $\forall L \in \text{Span}_K(\mathbf{D}), L(\mathbf{M}^{(\lambda\rho)}) = L^{[\lambda]}(\mathbf{M}^{(\lambda\rho)})$.

With the same notation of Prop. 16, a straightforward computation allows to verify that:

Lemma 18. *For $3 \le k \le n$*

$$L_{n+k}^{[k-1]} = \sum_{i=k}^{\frac{n+k}{2}} D(Y_i Y_{n+k-i}) + \sum_{j=k}^{n} d_{j2} D(Y_{k-2}Y_j)$$

$$+ \sum_{j=k-1}^{n} d_{j1} D(Y_{k-1}Y_j) + d_{k-1\ 1} D(Y_{k-1}^2).$$

Proposition 19. *Let $\mathbf{M}$ be the matrix of Proposition 16. Then $L_{n+k}^{[k-1]}(\mathbf{M}^{(\lambda\rho)}) = 0$ for $3 \le k \le n$, $\lambda \ge k - 1$.*

Proof. At this point it is just a question of explicitly computing $L_{n+k}^{[k-1]}(\mathbf{M}^{(\lambda\rho)})$, obtaining

$$\begin{cases} d_{k-1\ 1} + \sum_{j=k+1}^n d_{j1}c_{j+1-k} + d_{k1}d_{n1} - c_{n+2-k} - d_{k2} = 0 & \text{if } \lambda = k-1, \rho = k, \\ d_{\rho-1\ 1} + \sum_{j=\rho+1}^n d_{j1}c_{j+1-\rho} + d_{\rho1}d_{n1} - c_{n+2-\rho} - d_{\rho2} = 0 & \text{if } \lambda = k-1, \rho > k, \\ c_{n+k-\lambda-\rho+1} - c_{n+k-\lambda-\rho+1} = 0 & \text{if } \lambda > k-1, \end{cases}$$

where the expressions for $L_{n+k}^{[k-1]}(\mathbf{M}^{(k-1\ \rho)})$ are 0 because of the equalities (**) of Proposition 16. □

Corollary 20. *If $\mathfrak{q}$ is a primary ideal at the origin in l.g.p. of multiplicity $\mu = n+2+d$, $1 \le d \le n-1$, then it lies in a rational normal scroll of dimension d.*

Proof. In fact, in an appropriate frame of coordinates, the dual basis of $\mathfrak{q}$ is

$$\{\delta_0, \ldots, \delta_n, L_{n+1}, \ldots, L_{n+d+1}\}.$$

Let $\mathcal{S}$ be the ideal generated by the minors of the matrix containing only the last $n-d+1$ columns of $\mathbf{M}$; by Prop. 19, $\mathcal{S} \subset \mathfrak{q}$. □

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