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A CONVOLUTION ESTIMATE FOR A MEASURE ON A CURVE IN $\mathbb{R}^4$

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ABSTRACT. Let $\gamma(t) = (t, t^2, t^3, t^4)$ and fix an interval $I \subset \mathbb{R}$. If T is the operator on $\mathbb{R}^4$ defined by $Tf(x) = \int_I f(x - \gamma(t))\,dt$, then T maps $L^{\frac{5}{3}}(\mathbb{R}^4)$ into $L^2(\mathbb{R}^4)$.

INTRODUCTION

For $n \geq 2$, let $\gamma(t) = (t, t^2, \dots, t^n)$ for $t \in \mathbb{R}$. For an interval $I \subseteq \mathbb{R}$, define the operator T by

$$Tf(x) = \int_I f(x - \gamma(t))\,dt$$

for suitable functions f on $\mathbb{R}^n$. We are interested in determining the type set $\mathcal{T}$ of T — the set of points $(\frac{1}{p}, \frac{1}{q}) \in [0,1] \times [0,1]$ such that T maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. Let S be the closed segment, degenerate if $n = 2$, on the line $\frac{1}{p} - \frac{1}{q} = \frac{2}{n(n+1)}$ with endpoints $(\frac{n^2-n+2}{n^2+n}, \frac{n-1}{n+1})$ and $(\frac{2}{n+1}, \frac{2n-2}{n^2+n})$. Then (see §4 of [4]) $\mathcal{T}$ is contained in the closed convex hull of S and $\{(0,0),(1,1)\}$ if I is bounded, and in S if I is unbounded. If $n = 2$ or 3 these inclusions are equalities. We conjecture that this is true for any n. The main result in the case $n = 3$ is that T maps $L^{\frac{3}{2}}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$. This was proved in [5]. The method there was later used in [2], [8], [9], and [10] to treat more general curves in $\mathbb{R}^3$. The only nontrivial result for $n > 3$ is the fact, proved in [4], that the midpoint of S lies in $\mathcal{T}$. The purpose of this paper is to prove a partial result if $n = 4$.

Theorem 1. *Suppose $n = 4$. The segment on $\frac{1}{p} - \frac{1}{q} = \frac{1}{10}$ with endpoints $(\frac{1}{2}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{1}{2})$ lies in $\mathcal{T}$.*

Our approach here is different from those employed previously but, relying on the method of T^*T, it suffers from the same flaw as the method of [5]: the range space must be L^2.

Let P be the polynomial on $\mathbb{R}^4$ defined by

$$P(a,b,c,d) = 108a^2d^2 + 32b^3d - 108abcd + 27ac^3 - 9c^2b^2.$$

Theorem 2. *The function $|P|^{-\frac{1}{10}}$ is a Fourier multiplier of $L^{\frac{5}{3}}(\mathbb{R}^4)$ into $L^2(\mathbb{R}^4)$.*

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Theorem 3. *There is an absolute constant C such that*

$$\left| \int_I e^{i(at+bt^2+ct^3+dt^4)}\,dt \right| \le C|P(a,b,c,d)|^{-\frac{1}{10}}$$

for any $(a,b,c,d) \in \mathbb{R}^4$ and any interval $I \subseteq \mathbb{R}$.

Theorem 1 follows from Theorems 2 and 3 since

$$\widehat{Tf}(a,b,c,d) = \widehat{f}(a,b,c,d) \int_I e^{-i(at+bt^2+ct^3+dt^4)}\,\mathrm{d}t.$$

(The importance of L^2 as range space here is evident.) There are simpler analogues of Theorems 2 and 3 that yield a proof of the $L^{\frac{3}{2}}(\mathbb{R}^3) - L^2(\mathbb{R}^3)$ result as well: let

$$Q(a,b,c) = b^2 - 3ac.$$

Theorem 2′. *The function $|Q|^{-\frac{1}{4}}$ is a Fourier multiplier of $L^{\frac{3}{2}}(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$.*

Theorem 3′. *There is an absolute constant C such that*

$$\left| \int_I e^{i(at+bt^2+ct^3)}\,dt \right| \le C|Q(a,b,c)|^{-\frac{1}{4}}.$$

It seems likely that there are also analogues of Theorems 2 and 3 for $n \ge 5$ and that these will yield an analogue of Theorem 1: boundedness of T on the segment on $\frac{1}{p} - \frac{1}{q} = \frac{2}{n(n+1)}$ with endpoints $(\frac{1}{2}, \frac{1}{q})$ and $(\frac{1}{p}, \frac{1}{2})$. But there are some significant technical complications as n increases.

Theorems 3 and 3′ are proved in [7], which contains estimates for integrals

$$\int_I e^{ip(t)}\,dt$$

in case p is a polynomial with real coefficients. Most of this paper is devoted to the proof of Theorem 2. But we begin with a sketch of the (quite routine) proof of Theorem 2′, since it is based on the two principles, the method of T^*T and analytic interpolation, which yield Theorem 2. In what follows, the positive constant C may increase from line to line but will depend only on the parameters n, p, and q.

We sketch the proof of Theorem 2′. Suppose $1 \le p \le 2 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The "method of T^*T" is the statement that an operator T is bounded from L^p to L^2 if and only if T^*T is bounded from L^p to L^q. The proof is that both statements are equivalent to

$$|\langle T^*Tf, g\rangle| = |\langle Tf, Tg\rangle| \le \mathrm{C}\|f\|_p \|g\|_p.$$

Thus Theorem 2′ is equivalent to the statement that $|Q|^{-\frac{1}{2}}$ is a Fourier multiplier of $L^{\frac{3}{2}}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$. By a linear change of variables we can assume that $Q(a,b,c) = a^2 + b^2 - c^2$. Consider the analytic family of multipliers

$$m_z = (z+1)\left(z + \frac{3}{2}\right)|Q|^z, \quad -\frac{3}{2} \le \mathrm{Re}\ z \le 0.$$

If Re $z = 0$, these are bounded on $L^2(\mathbb{R}^3)$ with the multiplier norm growing polynomially in Im z. If Re $z = -\frac{3}{2}$, the Fourier transform formulas of [3] show that the L^∞ norms of $\widehat{m}_z$ (and so the $L^1(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ multiplier norms of m_z) grow polynomially in Im z. Thus Stein's theorem on analytic interpolation shows that $|Q|^{-\frac{1}{2}}$ is a Fourier multiplier of $L^{\frac{3}{2}}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$.

Proof of Theorem 2

We begin with a localization lemma, given in more generality that we really require. With a positive integer n fixed, let $M(p,q)$ denote the collection of Fourier multipliers from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. We will say that a function m on $\mathbb{R}^n$ belongs locally to $M(p,q)$ at $\xi \in \mathbb{R}^n$ if there is some function φ on $\mathbb{R}^n$ with $\varphi = 1$ in a neighborhood of ξ such that $m\varphi \in M(p,q)$.

Lemma. *Suppose $1 < p \leq 2 \leq q < \infty$ and m is homogeneous of degree $\frac{n}{q} - \frac{n}{p}$. If m belongs locally to $M(p,q)$ at each nonzero $\xi \in \mathbb{R}^n$, then $m \in M(p,q)$.*

The proof is an easy consequence of the Littlewood-Paley decomposition: Suppose Δ is the collection of rectangles effecting the standard dyadic decomposition of $\mathbb{R}^n$. If $\rho \in \Delta$, let S_ρ be the multiplier operator with symbol χ_ρ. Then the two estimates

$$\left(\sum_{\rho\in\Delta} \|S_\rho f\|_p^2\right)^{\frac{1}{2}} \leq \mathrm{C}\|f\|_p,$$

$$\|f\|_q \leq \mathrm{C}\left(\sum_{\rho\in\Delta} \|S_\rho f\|_q^2\right)^{\frac{1}{2}}$$

are well-known consequences of the Littlewood-Paley decomposition and Minkowski's inequality. Together they show that if S is the multiplier operator with symbol m, then the lemma will follow from the estimates, uniform in ρ,

$$\|SS_\rho f\|_q \leq \mathrm{C}\|S_\rho f\|_p. \tag{2.1}$$

(See [1] for an earlier application of this idea.)

It is a consequence of the locality hypothesis on m and a simple argument involving compactness and a partition of unity that there is a function φ on $\mathbb{R}^n$ satisfying

$$m\varphi \in M(p,q), \quad \varphi(\xi) = 1 \quad \text{if } 1 \leq |\xi| \leq \sqrt{n}.$$

Fix $\rho \in \Delta$ and let 2^j be the length of the longest side of ρ. Then $\varphi(\xi) = 1$ on the support of $\chi_\rho(2^j\xi)$. Letting $\|\cdot\|_{pq}$ denote the multiplier norm, it follows from $m\varphi \in M(p,q)$ that

$$\|m(\xi)\chi_\rho(2^j\xi)\|_{pq} \leq \mathrm{C}. \tag{2.2}$$

A homogeneity argument shows that

$$\|m(2^{-j}\xi)\chi_\rho(\xi)\|_{pq} = 2^{-j(\frac{n}{q}-\frac{n}{p})}\|m(\xi)\chi_\rho(2^j\xi)\|_{pq}.$$

Now (2.1) follows from (2.2) and the homogeneity hypothesis on m.

To apply the lemma to the proof of Theorem 2 we will take $n = 4$, $p = 2$, $q = \frac{5}{2}$, and $m = |P|^{-\frac{1}{10}}$. We must show that if $\xi_0 = (a_0, b_0, c_0, d_0) \neq 0$, then m belongs locally to $M(2, \frac{5}{2})$ at ξ_0. There are three cases: $P(\xi_0) \neq 0$, $P(\xi_0) = 0$ and $\operatorname{grad} P(\xi_0) \neq 0$, $P(\xi_0) = 0$ and $\operatorname{grad} P(\xi_0) = 0$. The first case is easy — if $P(\xi_0) \neq 0$, then m is bounded on some neighborhood of ξ_0. And any measurable function bounded in a neighborhood of ξ_0 belongs locally to $M(2,q)$ at ξ_0 for any $q \geq 2$.

The second and third cases follow the pattern of the proof of Theorem $2'$: roughly, if N is an appropriate neighborhood of ξ_0, we consider an analytic family of multipliers defined formally by

$$m_z(\xi) = p(z)|P(\xi)|^z \chi_{_N}(\xi)$$

for some polynomial p. If Re $z = 0$ (respectively, Re $z = -1$) the multiplier norms $\|m_z\|_{22}$ (respectively, $\|m_z\|_{1\infty}$) turn out to have polynomial growth in $|\mathrm{Im}\ z|$. So $m_{-\frac{2}{10}}$ is in $M(\frac{5}{3}, \frac{5}{2})$ by Stein's theorem on interpolation with analytic families. Thus

$$m\chi_{_N} \in M\left(2, \frac{5}{2}\right)$$

by the method of T^*T. The details of this argument are routine for the second case, more complicated in the third.

Let Σ be the surface in $\mathbb{R}^4$ defined by $P = 0$, and write $\mathrm{d}\sigma$ for surface area measure on Σ. In the second case ($P(\xi_0) = 0$, $\mathrm{grad}\, P(\xi_0) \neq 0$) choose a small neighborhood $\widetilde{N}$ of ξ_0 in Σ and a small positive number δ such that the mapping

$$\xi = \sigma + t\,\mathrm{grad}\, P(\sigma), \qquad \sigma \in \widetilde{N},\ t \in (-\delta, \delta),$$

gives a one-to-one parametrization of some neighborhood N of ξ_0 on which

$$\frac{\mathrm{d}\xi}{\mathrm{C}} \leq \mathrm{d}\sigma\, \mathrm{d}t \leq \mathrm{C}\, \mathrm{d}\xi,$$

$$\frac{1}{\mathrm{C}} \leq |\mathrm{grad}\, P(\sigma)|, \qquad \text{and} \tag{2.3}$$

$$\frac{|t|}{\mathrm{C}} \leq |P(\xi)|.$$

Consider the family of operators defined, for $x \in \mathbb{R}^4$, by

$$T_z f(x) = (z+1) \int_{\widetilde{N}} \int_{-\delta}^{\delta} \widehat{f}(\sigma + t\,\mathrm{grad}\, P(\sigma)) e^{ix\cdot(\sigma + t\,\mathrm{grad}\, P(\sigma))} |t|^z\, \mathrm{d}t\, \mathrm{d}\sigma.$$

From the conditions (2.3) it follows that we need only check that the quantities defined formally by

$$\sup_{x \in \mathbb{R}^4} \left| (z+1) \int_{\widetilde{N}} \int_{-\delta}^{\delta} e^{ix\cdot(\sigma + t\,\mathrm{grad}\, P(\sigma))} |t|^z\, \mathrm{d}t\, \mathrm{d}\sigma \right|$$

have polynomial growth in Im z if Re $z = -1$. Since

$$\int_{\widetilde{N}} \mathrm{d}\sigma < \infty$$

($\widetilde{N}$ is small), this follows from

$$\sup_{u \in \mathbb{R}} \left| \int_0^{\delta} u e^{iut} t^{is}\, \mathrm{d}t \right| \leq \mathrm{C}(1 + |s|),$$

which is a consequence of van der Corput's lemma.

Consider now the third case: $\xi_0 = (a_0, b_0, c_0, d_0) \neq 0$, $P(\xi_0) = 0$, $\operatorname{grad} P(\xi_0) = 0$. It is easy to check that not both a_0 and d_0 can be 0. We will assume $a_0 \neq 0$, the other case being similar. Let $N = \{\xi : |\xi - \xi_0| < \frac{|a_0|}{2}\}$, so that $|a| \geq \frac{|a_0|}{2}$ if $(a, b, c, d) \in N$. The equation $P(a, b, c, d) = 0$ is a quadratic equation in d, and solving it for d yields functions d_+ and d_- defined by

$$d_\pm(a, b, c) = \frac{27abc - 8b^3 \pm (4b^2 - 9ac)^{\frac{3}{2}}}{54a^2}.$$

Additionally, put

$$d_0(a, b, c) = \frac{27abc - 8b^3}{54a^2}.$$

Define $N_1 = N \cap \{9ac - 4b^2 < 0\}$ and $N_2 = N \cap \{9ac - 4b^2 > 0\}$, and let B_1 (respectively, B_2) be the projection of N_1 (respectively, N_2) onto $\{c = d = 0\}$. We start by recording some estimates for $|P|$ on N_1 and N_2.

A little algebra shows that if $a \neq 0$ then

$$P(a, b, c, d) = 108a^2(d - d_0)^2 + \frac{(9ac - 4b^2)^3}{27a^2}.$$

Since $\frac{|a_0|}{2} \leq |a| \leq \mathrm{C}$ on N, it follows that

$$\frac{|d - d_0| \cdot |9ac - 4b^2|^{\frac{3}{2}}}{\mathrm{C}} \leq |P(a, b, c, d)| \quad \text{if } (a, b, c, d) \in N_2.$$

If $(a, b, c, d) \in N_1$, then the estimates are

$$\frac{|d - d_+| \cdot |9ac - 4b^2|^{\frac{3}{2}}}{\mathrm{C}} \leq |P(a, b, c, d)| \quad \text{if } d_0 \leq d,$$

$$\frac{|d - d_-| \cdot |9ac - 4b^2|^{\frac{3}{2}}}{\mathrm{C}} \leq |P(a, b, c, d)| \quad \text{if } d \leq d_0.$$

For $j = -, 0, +$ define

$$\rho_j(a, b, c, d) = |d - d_j| \cdot |9ac - 4b^2|^{\frac{3}{2}}.$$

Then it is enough to show that

$$\rho_0^{-\frac{1}{10}} \chi_{B_2}, \quad \rho_+^{-\frac{1}{10}} \chi_{B_1}, \quad \text{and} \quad \rho_-^{-\frac{1}{10}} \chi_{B_1}$$

are in $M(2, \frac{5}{2})$. In accordance with our general plan (the method of T^*T combined with analytic interpolation) it is enough to show that the L^∞ norms of the inverse Fourier transforms of (the analytic continuations of)

$$(z+1)\left(z + \frac{2}{3}\right) \rho_0^z \chi_{B_2}, \quad (z+1)\left(z + \frac{2}{3}\right) \rho_+^z \chi_{B_1}, \quad (z+1)\left(z + \frac{2}{3}\right) \rho_-^z \chi_{B_1}$$

have polynomial growth in $|\mathrm{Im}\ z|$ if $\mathrm{Re}\ z = -1$. We will write s for $\mathrm{Im}\ z$.

In the first case the quantity to estimate is, formally,

$$\int_{B_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x_1 a + x_2 b + x_3 c + x_4 d)} \frac{\mathrm{d}d}{|d - d_0|^{1-is}} \frac{\mathrm{d}c}{|9ac - 4b^2|^{3(1-is)/2}} \, \mathrm{d}(b, a). \tag{2.4}$$

(The quantities above are defined by the analytic continuations of certain integrals. We maintain the integral notation only for convenience.)

Using the equation (see p. 359 of [3])

$$\int_{-\infty}^{\infty} e^{ix\sigma}|x|^{\lambda}\,\mathrm{d}x = -2\sin\frac{\lambda\pi}{2}\Gamma(\lambda+1)|\sigma|^{-\lambda-1} \qquad (\lambda \neq -1,-3,\dots), \tag{2.5}$$

we see that it is enough to estimate

$$\int_{B_2}\int_{-\infty}^{\infty} e^{i(x_1a+x_2b+x_3c+x_4d_0)}\frac{\mathrm{d}c}{|9ac-4b^2|^{3(1-is)/2}}\,\mathrm{d}(b,a).$$

Making the change of variable $C = 9ac - 4b^2$, we examine

$$\int_{B_2}\int_{-\infty}^{\infty} \exp\left\{i\left[x_1a + x_2b + \frac{4x_3}{9a}b^2 + \frac{2x_4}{27a^2}b^3 + \left(\frac{x_3}{9a} + x_4\frac{b}{18a^2}\right)C\right]\right\} \cdot\frac{\mathrm{d}C}{|C|^{3(1-is)/2}}\,\frac{\mathrm{d}(b,a)}{|a|}.$$

With $p_a(b) = x_1a + x_2b + \frac{4x_3}{9a}b^2 + \frac{2x_4}{27a^2}b^3$, use (2.5) again to obtain

$$\int_{B_2} \exp\{ip_a(b)\}\left|\frac{p_a''(b)}{8}\right|^{\frac{1}{2}(1-3is)}\mathrm{d}b\,\frac{\mathrm{d}a}{|a|}.$$

For each a, the integral on b is over some interval and so, by [6], is bounded by $\mathrm{C}(1+|s|)^{\frac{1}{2}}$. The integral on a is over a compact interval not containing 0, and so the supremum over $(x_1,x_2,x_3,x_4)\in\mathbb{R}^4$ of the absolute value of (2.4) grows only polynomially in $|s|$.

The estimates for $(z+1)(z+\frac{2}{3})\rho_+^z\chi_{B_1}$ and $(z+1)(z+\frac{2}{3})\rho_-^z\chi_{B_1}$ are similar to each other but a little more complicated than that for $(z+1)(z+\frac{2}{3})\rho_0^z\chi_{B_2}$. It is enough to estimate

$$\int_{I_a}\int_{-\infty}^{\infty} e^{i(x_1a+x_2b+x_3c+x_4d_+)}\frac{\mathrm{d}c}{|9ac-4b^2|^{3(1-is)/2}}\,\mathrm{d}b,$$

where I_a is an interval depending on a. The change of variable $C = 9ac - 4b^2$ leads now to

$$\int_{I_a}\int_{-\infty}^{\infty} \exp\left\{i\left[x_1a + x_2b + \frac{4x_3b^2}{9a} + \frac{2x_4b^3}{27a^2} + \left(\frac{x_3}{9a} + x_4\frac{b}{18a^2}\right)C + \frac{x_4}{54a^2}C^{\frac{3}{2}}\right]\right\}\frac{\mathrm{d}C}{|C|^{3(1-is)/2}}\,\mathrm{d}b. \tag{2.6}$$

Replacing b by $|x_4|^{-\frac{1}{3}}b$ and C by $|x_4|^{-\frac{2}{3}}C$ allows us to assume, without loss of generality, that $x_4 = 1$. The integration on b is over a new interval I_a', and x_2 and x_3 must be replaced by new values x_2' and x_3'. Writing $p(b) = x_1a + x_2'b + \frac{4x_3'}{9a}b^2 + \frac{2}{27a^2}b^3$, (2.6) becomes

$$\int_{I_a'}\int_{-\infty}^{\infty} \exp\left\{i\left(p(b)+\frac{Cp''(b)}{8}+\frac{C^{\frac{3}{2}}}{54a^2}\right)\right\}\frac{\mathrm{d}C}{|C|^{3(1-is)/2}}\mathrm{d}b. \tag{2.7}$$

The equality

$$\exp\left\{i\left(p(b)+\frac{Cp''(b)}{8}+\frac{C^{\frac{3}{2}}}{54a^2}\right)\right\}$$
$$=\exp\left\{i\left(p(b)+\frac{Cp''(b)}{8}\right)\right\}\left\{1+\left[\exp\left(\frac{iC^{\frac{3}{2}}}{54a^2}\right)-1\right]\right\}$$

splits (2.7) into the sum of two terms. We have already treated the first of these. The second can be written

$$\int_{-\infty}^{\infty}\int_{I_a'} \exp\left\{i\left(p(b)+\frac{Cp''(b)}{8}\right)\right\}db\left[\exp\left(\frac{iC^{\frac{3}{2}}}{54a^2}\right)-1\right]\frac{\mathrm{d}C}{|C|^{3(1-is)/2}}. \tag{2.8}$$

(The Fubini-like interchange here can be justified by uniqueness of analytic continuation.) The phase function $p(b)+\frac{Cp''(b)}{8}$ is a third degree polynomial with leading term $\frac{2b^3}{27a^2}$. Since a lies in a bounded interval, van der Corput's lemma bounds the absolute value of the inner term. Since a is bounded away from 0, (2.8) is bounded as well. This completes the proof of Theorem 2.

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