

## POLYNOMIAL CONTINUITY ON $\ell_1$

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ABSTRACT. A mapping between Banach spaces is said to be polynomially continuous if its restriction to any bounded set is uniformly continuous for the weak polynomial topology. A Banach space  $X$  has property (RP) if given two bounded sequences  $(u_j), (v_j) \subset X$ , we have that  $Q(u_j) - Q(v_j) \rightarrow 0$  for every polynomial  $Q$  on  $X$  whenever  $P(u_j - v_j) \rightarrow 0$  for every polynomial  $P$  on  $X$ ; i.e., the restriction of every polynomial on  $X$  to each bounded set is uniformly sequentially continuous for the weak polynomial topology. We show that property (RP) does not imply that every scalar valued polynomial on  $X$  must be polynomially continuous.

Throughout,  $X$  and  $Y$  are Banach spaces,  $X^*$  the dual of  $X$ ,  $B_X$  its closed unit ball,  $S_X$  its unit sphere, and  $\mathbf{N}$  the set of natural numbers. Given  $k \in \mathbf{N}$ , we denote by  $\mathcal{P}({}^kX, Y)$  the space of all  $k$ -homogeneous (continuous) polynomials from  $X$  into  $Y$ ;  $\mathcal{L}_s({}^kX, Y)$  is the space of all (continuous) symmetric  $k$ -linear mappings from  $X^k := X \times \cdots \times X$  into  $Y$ . Whenever  $Y$  is omitted, it is understood to be the scalar field  $\mathbf{K}$  (real  $\mathbf{R}$  or complex  $\mathbf{C}$ ). We identify  $\mathcal{P}({}^0X) = \mathbf{K}$ , and denote  $\mathcal{P}(X) := \sum_{k=0}^{\infty} \mathcal{P}({}^kX)$ . For the general theory of polynomials on Banach spaces, we refer to [6]. As usual,  $e_n$  stands for the sequence  $(0, \dots, 0, 1, 0, \dots)$  with 1 in the  $n$ th position.

To each polynomial  $P \in \mathcal{P}({}^kX, Y)$  we can associate a unique symmetric  $k$ -linear mapping  $\hat{P} \in \mathcal{L}_s({}^kX, Y)$  so that  $P(x) = \hat{P}(x, \dots, x)$  for all  $x \in X$ , and a (bounded linear) operator  $T_P : X \rightarrow \mathcal{L}_s({}^{k-1}X, Y)$  given by

$$T_P(x)(x_1, \dots, x_{k-1}) = \hat{P}(x, x_1, \dots, x_{k-1}).$$

Following [1], we say that a mapping  $f : X \rightarrow Y$  is *polynomially continuous* ( $P$ -continuous, for short) if, for every  $\epsilon > 0$  and bounded  $B \subset X$ , there are a finite set  $\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$  and  $\delta > 0$  so that  $\|f(x) - f(y)\| < \epsilon$  whenever  $x, y \in B$  satisfy  $|P_j(x - y)| < \delta$  ( $1 \leq j \leq n$ ).

Clearly, the definition may be restated assuming that the polynomials  $P_1, \dots, P_n$  are homogeneous.

Suppose we require the polynomials  $\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$  in the above definition to be of degree one, i.e., to be continuous linear forms on  $X$ . Then we obtain that  $f$  is weakly uniformly continuous on bounded subsets, a notion that has been studied

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by many authors (see [1]). Since an operator is compact if and only if it is weakly (uniformly) continuous on bounded sets [2, Proposition 2.5], every compact operator is  $P$ -continuous. If a polynomial is weakly (uniformly) continuous on bounded sets (such as every scalar valued polynomial on  $c_0$ ), then it is clearly  $P$ -continuous.

We shall need the following result:

**Proposition 1.** *A polynomial  $P$  is  $P$ -continuous if and only if so is the associated operator  $T_P$ .*

*Proof.* Suppose  $P \in \mathcal{P}({}^kX, Y)$  is  $P$ -continuous. Given  $\epsilon > 0$ , we can find  $\delta > 0$  and  $\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$  so that  $\|P(x) - P(y)\| < \epsilon$  whenever  $|P_j(x - y)| < \delta$  for all  $1 \leq j \leq n$  and  $x, y \in B_X$ .

Assume  $x, y$  satisfy the above conditions, and  $z_1, \dots, z_{k-1} \in B_X$ . The polarization formula [6, Theorem 1.10] yields:

$$\begin{aligned} & (T_P(x) - T_P(y))(z_1, \dots, z_{k-1}) \\ &= \hat{P}(x, z_1, \dots, z_{k-1}) - \hat{P}(y, z_1, \dots, z_{k-1}) \\ &= \frac{k^k}{k!2^k} \sum_{\epsilon_j = \pm 1} \epsilon_1 \cdots \epsilon_k \left[ P \left( \frac{\epsilon_1 x + \epsilon_2 z_1 + \cdots + \epsilon_k z_{k-1}}{k} \right) \right. \\ & \qquad \qquad \qquad \left. - P \left( \frac{\epsilon_1 y + \epsilon_2 z_1 + \cdots + \epsilon_k z_{k-1}}{k} \right) \right]. \end{aligned}$$

Assuming that every  $P_j$  is homogeneous, we have

$$\begin{aligned} & \left| P_j \left( \frac{\epsilon_1 x + \epsilon_2 z_1 + \cdots + \epsilon_k z_{k-1}}{k} - \frac{\epsilon_1 y + \epsilon_2 z_1 + \cdots + \epsilon_k z_{k-1}}{k} \right) \right| \\ & < |P_j(\epsilon_1 x - \epsilon_1 y)| \\ & = |P_j(x - y)| \\ & < \delta \end{aligned}$$

for  $1 \leq j \leq n$ , and so

$$\|T_P(x) - T_P(y)\| \leq \frac{\epsilon k^k}{k!}.$$

Conversely, let  $T_P$  be  $P$ -continuous. For  $0 < \epsilon < 1$ , there is  $\delta > 0$  and  $\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$  so that  $\|T_P(x) - T_P(y)\| < \epsilon$ , whenever  $|P_j(x - y)| < \delta$  for any  $1 \leq j \leq n$  and  $x, y \in B_X$ . For such  $x, y$  we have

$$\begin{aligned} & \|P(x) - P(y)\| \\ & \leq \|\hat{P}(x, \dots, x) - \hat{P}(x, y, x, \dots, x)\| + \|\hat{P}(x, y, x, \dots, x) - \hat{P}(x, y, y, x, \dots, x)\| \\ & \quad + \cdots + \|\hat{P}(x, y, \dots, y) - \hat{P}(y, \dots, y)\| \\ & = \|(T_P(x) - T_P(y))(x, \dots, x)\| + \|(T_P(x) - T_P(y))(x, y, x, \dots, x)\| \\ & \quad + \cdots + \|(T_P(x) - T_P(y))(y, \dots, y)\| \\ & < k\epsilon, \end{aligned}$$

and the proof is complete. □

We say that a net  $(x_\alpha) \subset X$  converges to  $x$  in the *weak polynomial topology* (*pw-topology*, for short) [3, §6] if for every  $P \in \mathcal{P}(X)$  we have  $P(x_\alpha) \rightarrow P(x)$ .

It is clear that a mapping  $f : X \rightarrow Y$  is *pw*-continuous on bounded sets if and only if for every  $x \in X$ ,  $\epsilon > 0$  and bounded  $B \subset X$  with  $x \in B$ , there are  $\delta > 0$  and

$\{P_1, \dots, P_n\} \subset \mathcal{P}(X)$  so that we have  $\|f(x) - f(y)\| < \epsilon$  whenever  $|P_j(x - y)| < \delta$  for  $1 \leq j \leq n$  and  $y \in B$ . Obviously, an operator is  $P$ -continuous if and only if it is  $pw$ -continuous on bounded sets.

We now relate the  $P$ -continuity with property (RP) of Aron, Choi and Llavona [1]. We say that  $X$  has *property (RP)* if given two bounded sequences  $(u_j)$  and  $(v_j)$  in  $X$ , we have that  $Q(u_j) - Q(v_j) \rightarrow 0$  for every  $Q \in \mathcal{P}(X)$  whenever  $P(u_j - v_j) \rightarrow 0$  for every  $P \in \mathcal{P}(X)$ .

Every superreflexive space and every space with the DPP not containing  $\ell_1$  have property (RP) [1]. Clearly, if every scalar valued (continuous) polynomial on  $X$  is  $P$ -continuous, then  $X$  has property (RP). It is proved in [1] that  $C[0, 1]$ ,  $L_1[0, 1]$  and  $L_\infty[0, 1]$  do not satisfy property (RP), and that there are 3-homogeneous polynomials on the spaces  $C[0, 1]$  and  $L_\infty[0, 1]$  which are not  $P$ -continuous. Similarly, there is a non- $P$ -continuous 2-homogeneous polynomial on  $L_1[0, 1]$ .

It is natural to ask whether property (RP) implies that every scalar valued polynomial is  $P$ -continuous. We show that the answer is no by giving examples of polynomials on  $\ell_1$  which are not  $P$ -continuous. We first need to construct a  $pw$ -null net in the sphere of  $\ell_1$ . We need a previous lemma.

**Lemma 2.** *Let  $U$  be a weak zero neighbourhood in  $\ell_1$ . Then, for each  $m \in \mathbf{N}$  we can find  $x = (x_n) \in S_{\ell_1} \cap U$  and  $r > m$  so that  $x_n = 0$  whenever  $n < m$  and  $n > r$ .*

*Proof.* We can find  $\xi_1, \dots, \xi_k \in B_{\ell_\infty}$  and  $\epsilon > 0$  such that

$$U \supseteq \{x \in \ell_1 : |\xi_j(x)| < \epsilon \text{ for } 1 \leq j \leq k\} .$$

Let  $\xi_j = (\xi_j^n)_{n=1}^\infty$ . There is an infinite set  $A \subset \mathbf{N}$  so that  $|\xi_j^p - \xi_j^q| < 2\epsilon$  whenever  $1 \leq j \leq k$  and  $p, q \in A$ . Fix  $p, q \in A$  ( $m \leq p < q$ ), and set  $x := (e_p - e_q)/2$  and  $r = q$ . Then  $|\xi_j(x)| = |\xi_j^p - \xi_j^q|/2 < \epsilon$  for  $1 \leq j \leq k$ , and the proof is complete.  $\square$

The following two results use the idea of [4].

**Lemma 3.** *Let  $\mathcal{F}$  be a finite family of continuous symmetric multilinear forms on  $\ell_1$ ,  $\epsilon > 0$  and  $N \geq 1$ . Then there exist  $x_1, \dots, x_N \in S_{\ell_1}$ , with disjoint supports, such that  $|F(x_{i_1}, \dots, x_{i_m})| < \epsilon$  whenever  $F \in \mathcal{F}$  is an  $m$ -form and  $i_1, \dots, i_m$  are distinct indices between 1 and  $N$ .*

*Proof.* Since each  $F \in \mathcal{F}$  is symmetric, it is enough to obtain the estimate when  $i_1 < \dots < i_m$ .

By Lemma 2, we can find  $n_1 \in \mathbf{N}$  and  $x_1 \in S_{\ell_1}$ , having all but the first  $n_1$  coordinates equal to zero, so that  $|F(x_1)| < \epsilon$  for all  $F \in \mathcal{F} \cap \ell_1^*$ . Again by Lemma 2, we can choose  $n_2 \in \mathbf{N}$  and  $x_2 \in S_{\ell_1}$  having disjoint support with  $x_1$  and all but the first  $n_2$  coordinates equal to zero, so that  $|F(x_2)| < \epsilon$  for all  $F \in \mathcal{F} \cap \ell_1^*$ , and  $|F(x_1, x_2)| < \epsilon$  for all  $F \in \mathcal{F} \cap \mathcal{L}_s(2\ell_1)$ . In this way, we obtain  $x_j$ 's with disjoint supports, so that  $|F(x_{i_1}, \dots, x_{i_m})| < \epsilon$  for all  $F \in \mathcal{F} \cap \mathcal{L}_s(m\ell_1)$  and all  $i_1 < \dots < i_m$ .  $\square$

**Theorem 4.** *There is a  $pw$ -null net in  $S_{\ell_1}$ .*

*Proof.* It is enough to show that for every finite family  $\mathcal{F} \subset \mathcal{P}(\ell_1)$  and  $\epsilon > 0$ , there is an  $x \in S_{\ell_1}$  so that  $|P(x)| < \epsilon$  for all  $P \in \mathcal{F}$ .

Fix  $N$  large, choose  $x_1, \dots, x_N \in S_{\ell_1}$  with disjoint supports satisfying the conditions of Lemma 3 for the family  $\{\hat{P} : P \in \mathcal{F}\}$  of symmetric multilinear forms, and set

$$x := \frac{1}{N} (x_1 + \dots + x_N) \in S_{\ell_1} .$$

If  $P \in \mathcal{F} \cap \mathcal{P}(^m \ell_1)$ , we write

$$P(x) = \frac{1}{N^m} \sum_{i_1, \dots, i_m=1}^N \hat{P}(x_{i_1}, \dots, x_{i_m}) = \Sigma_1 + \Sigma_2,$$

where  $\Sigma_1$  is the sum over  $m$ -tuples of distinct indices, and  $\Sigma_2$  is the sum over the remaining indices.

By Lemma 3,  $|\Sigma_1| < \epsilon/2$ . Since there are  $N^m - N(N-1)\cdots(N-m+1)$  summands in  $\Sigma_2$ , we obtain

$$|\Sigma_2| \leq \left[ 1 - \left( 1 - \frac{1}{N} \right) \cdots \left( 1 - \frac{m-1}{N} \right) \right] \cdot \|\hat{P}\| < \frac{\epsilon}{2},$$

for  $N$  large enough. □

As a consequence, if  $X$  contains a copy of  $\ell_1$ , then the unit sphere of  $X$  contains a  $pw$ -null net as well. We now give the main result.

**Theorem 5.** *For every  $k \in \mathbf{N}$  ( $k \geq 2$ ), there is a  $k$ -homogeneous scalar valued polynomial on  $\ell_1$  which is not  $P$ -continuous.*

*Proof.* Suppose first that  $k = 2$  and  $\ell_1$  is constructed over the real numbers. We need a sequence  $(x_j) \subset \ell_\infty$ , equivalent to the  $\ell_1$ -basis, such that  $x_i^j = x_j^i$  for all  $i, j \in \mathbf{N}$ , where  $x_j^i := x_j(e_i)$  (then we say that the sequence is *symmetric*).

We select a Rademacher-like sequence  $(y_j) \subset \ell_\infty$ , taking  $y_1 := (1, -1, 1, -1, \dots)$ , and letting  $y_j$  be the sequence consisting of infinitely many times the following block of  $2^j$  integers:

$$1, (2^{j-1}), 1, -1, (2^{j-1}), -1.$$

Clearly,  $(y_j)$  is 1-equivalent to the unit vector basis of  $\ell_1$ ; i.e., for every finite set of real numbers  $\alpha_1, \dots, \alpha_n$ , we have

$$(1) \quad \sum_{j=1}^n |\alpha_j| = \left\| \sum_{j=1}^n \alpha_j y_j \right\|_\infty.$$

If we take  $x_1 := y_1$  and, for  $j > 1$ , modify the first  $j-1$  coordinates of  $y_j$  in the obvious way, then we get a symmetric sequence  $(x_j)$  which is still 1-equivalent to the unit vector basis of  $\ell_1$ .

Now, define an operator  $T : \ell_1 \rightarrow \ell_\infty$  by  $T(e_j) := x_j$ . Since  $T$  is an embedding, we conclude from Theorem 4 that it is not  $P$ -continuous. Therefore, by Proposition 1, the 2-homogeneous polynomial  $P : \ell_1 \rightarrow \mathbf{R}$  given by  $P(y) = (T(y))(y)$  for  $y \in \ell_1$  is not  $P$ -continuous.

The same sequence can be used in the complex case, since, for every finite set of complex numbers  $\alpha_1, \dots, \alpha_n$ , we have (see [5, XI, Proposition 4])

$$(2) \quad \sum_{j=1}^n |\alpha_j| \leq 4 \left\| \sum_{j=1}^n \alpha_j y_j \right\|_\infty.$$

The sequence  $(y_j)$  will be used in the case  $k > 2$  as well. Letting

$$A_j(e_{i_2}, \dots, e_{i_k}) := \begin{cases} y_j^{i_2 + \dots + i_k} & \text{if } j \leq \min\{i_2, \dots, i_k\}, \\ y_{i_r}^{j+i_2+\dots+i_{r-1}+i_{r+1}+\dots+i_k} & \text{if } i_r = \min\{i_2, \dots, i_k\} < j, \end{cases}$$

we obtain  $A_j \in \mathcal{L}_s(k-1\ell_1)$ . Moreover, the sequence  $(A_j)$  is equivalent to the unit vector basis of  $\ell_1$ . Indeed, given real numbers  $\alpha_1, \dots, \alpha_n$ , using (1), choose  $i_2, \dots, i_k \in \mathbf{N}$  such that  $n \leq \min\{i_2, \dots, i_k\}$  and

$$\sum_{j=1}^n |\alpha_j| = \left| \sum_{j=1}^n \alpha_j y_j^{i_2 + \dots + i_k} \right| = \left\| \sum_{j=1}^n \alpha_j A_j \right\|.$$

Note that the sequence  $(A_j)$  is symmetric in the sense that  $A_{i_1}(e_{i_2}, \dots, e_{i_k})$  is invariant under permutation of the indices  $i_1, \dots, i_k$ . In the complex case we proceed similarly, using (2) in place of (1). In both cases we define  $T : \ell_1 \rightarrow \mathcal{L}_s(k-1\ell_1)$  by  $T(e_j) = A_j$ . Then the polynomial  $P \in \mathcal{P}(k\ell_1)$  given by

$$P(\alpha) := \sum_{i_1, \dots, i_k=1}^{\infty} \alpha_{i_1} \cdots \alpha_{i_k} A_{i_1}(e_{i_2}, \dots, e_{i_k}), \quad \text{for } \alpha = (\alpha_j)_{j=1}^{\infty} \in \ell_1,$$

is not  $P$ -continuous, since the associated operator  $T$  is an isomorphism.  $\square$

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