

AN INVARIANCE OF DOMAIN RESULT
FOR MULTI-VALUED MAXIMAL MONOTONE
OPERATORS WHOSE DOMAINS DO NOT NECESSARILY
CONTAIN ANY OPEN SETS

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ABSTRACT. Let X be a real, reflexive, locally uniformly convex Banach space with X^* locally uniformly convex. Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a maximal monotone operator and $G \subset X$ open and bounded. Assume that $M \subset X^*$ is pathwise connected and such that $T(D(T) \cap G) \cap M \neq \emptyset$ and $\overline{T(D(T) \cap \partial G)} \cap M = \emptyset$. Then $M \subset T(D(T) \cap G)$. If, moreover, T is of type (S) on ∂G , then $\overline{T(D(T) \cap \partial G)}$ may be replaced above by $T(D(T) \cap \partial G)$. The significance of this result lies in the fact that it holds for multi-valued mappings T which do not have to satisfy $\text{int}D(T) \neq \emptyset$. It has also been used in this paper in order to establish a general “invariance of domain” result for maximal monotone operators, and may be applied to a greater variety of problems involving partial differential equations. No degree theory has been used. In addition to the above, necessary and sufficient conditions are given for the existence of a zero (in an open and bounded set G) of a completely continuous perturbation $T + C$ of a maximal monotone operator T such that $T + C$ is locally monotone on G .

1. INTRODUCTION AND PRELIMINARIES

In what follows, the symbol X stands for a real Banach space with norm $\|\cdot\|$ and (normalized) duality mapping J . Unless otherwise stated, we shall assume that X is reflexive and that X and X^* have been renormed so that they are locally uniformly convex. This implies that the duality mapping J is single-valued, monotone (see definition below) and bicontinuous.

The symbols $\langle x^*, x \rangle$ and $\langle x, x^* \rangle$ denote the value of the functional $x^* \in X^*$ at $x \in X$. We denote by $D(T)$, $R(T)$ and $G(T)$ the effective domain, the range and the graph of a mapping $T : X \rightarrow 2^{X^*}$, respectively. We have $D(T) = \{x \in X : Tx \neq \emptyset\}$ and $G(T) = \{(x, x^*) : x \in D(T), x^* \in Tx\}$. An operator $T : X \supset D(T) \rightarrow 2^{X^*}$ is said to be “injective” on $G \subset X$ if $x, y \in D(T) \cap G$ and $Tx \cap Ty \neq \emptyset$ imply $x = y$. In what follows, “continuous” means “strongly continuous” and the symbol “ \rightarrow ” (“ \dashrightarrow ”) means strong (weak) convergence. The symbol R (R_+) stands for the set $(-\infty, \infty)$ ($[0, \infty)$) and the symbols ∂D , $\text{int}D$, \overline{D} denote the strong boundary, interior and closure of the set D , respectively. An operator $T : X \supset D(T) \rightarrow Y$,

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with Y another real Banach space, is “bounded” if it maps bounded subsets of $D(T)$ onto bounded sets. It is “compact” if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets. It is “completely continuous” if it is continuous from the weak topology of $D(T)$ to the strong topology of Y . It is called “demicontinuous” if it is continuous from the strong topology of $D(T)$ to the weak topology of Y .

An operator $T : X \supset D(T) \rightarrow 2^{X^*}$ is “monotone” if

$$(*) \quad \langle u^* - v^*, x - y \rangle \geq 0 \text{ for every } x, y \in D(T) \text{ and every } u^* \in Tx, v^* \in Ty.$$

A monotone operator T is “strongly monotone” if 0 in the right-hand side of $(*)$ is replaced by $\alpha\|x - y\|^2$, where $\alpha > 0$ is a fixed constant. A monotone operator T is called “maximal monotone” if its graph $G(T)$ is a maximal monotone subset of $X \times X^*$. A monotone operator T is maximal monotone if and only if $R(T + \lambda J) = X^*$ for all $\lambda > 0$. An operator $T : X \supset D(T) \rightarrow 2^{X^*}$ is said to be of “type (S) ” on the set $E \subset X$, if for every sequence $\{x_n\} \subset D(T) \cap E$ with $x_n \rightarrow x_0 \in X$ and

$$\lim_{n \rightarrow \infty} \langle v_n^*, x_n - x_0 \rangle = 0,$$

for some $v_n^* \in Tx_n$, we have $x_n \rightarrow x_0$.

We denote by $\text{co}A$ the convex hull of a set A . Two sets $A \subset X$, $B \subset X$ are called “separated” if $(\overline{A} \cap B) \cup (A \cap \overline{B}) = \emptyset$. A subset M of a Banach space X is called “pathwise connected” if for every $x, y \in M$ there exists a continuous function $s : [0, 1] \rightarrow M$ such that $s(0) = x$ and $s(1) = y$. The function s is called a “path.” A pathwise connected set M , associated with the norm topology, is connected. A set M , associated with the norm topology, is pathwise connected if and only if it is connected and every point in M has a pathwise connected neighborhood. For these facts, the reader is referred to Dugundji [5, pp. 114-116].

Let G be an open and bounded subset of X . The purpose of this paper is to give a general theorem, Theorem 1, according to which a pathwise connected set $M \subset X^*$ lies in the range of a maximal monotone operator T . The conditions that the set M has to satisfy are: $T(D(T) \cap G) \cap M \neq \emptyset$ and $\overline{T(D(T) \cap \partial G)} \cap M = \emptyset$. The conclusion is: $M \subset T(D(T) \cap G)$. Results of this type were given recently by Guan [6] (for demicontinuous monotone operators with G convex), Park [15] (for demicontinuous monotone operators in Hilbert spaces) and Kartsatos [13] (for demicontinuous monotone operators with G not necessarily convex). The main characteristic of our result is the fact that $D(T)$ does not necessarily contain an open set, which results in the absence of any continuity property imposed upon the operator T . This result, Theorem 1, is then used in order to establish a result, Proposition 1, according to which a maximal monotone operator T which is locally injective and of type (S) on an open set G has the property that $T(D(T) \cap G)$ is open. Finally, a result is given, Proposition 2, which says that the Leray-Schauder condition is necessary and sufficient for the existence of a zero of the sum $T + C$, where T is maximal monotone and C is completely continuous. We do require the additional assumption that $T + C$ is locally monotone on a bounded and open set G .

Lemma A. *Let $\{s_n\}$ be a nonincreasing sequence of positive numbers, with $s_n \rightarrow 0$ as $n \rightarrow \infty$, and $\{x_n\} \subset X$. Assume that for all m, n we have*

$$\langle s_n Jx_n - s_m Jx_m, x_n - x_m \rangle \leq 0.$$

Then $\{\|x_n\|\}$ is a nondecreasing sequence. If it is bounded, then $x_n \rightarrow$ (some) $x \in X$.

This lemma follows easily from Lemma 1.4 of [2].

For facts about the theory of monotone operators the reader is referred to Barbu [1], Browder [3], Pascali and Sburlan [16] and Zeidler [19]. For results related to the ones in this paper, we refer to the papers [6]–[15] and some of the references therein.

2. MAIN RESULTS

Theorem 1. *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and let G be an open and bounded subset of X . Assume that $M \subset X^*$ is pathwise connected and such that $T(D(T) \cap G) \cap M \neq \emptyset$ and $\overline{T(D(T) \cap \partial G)} \cap M = \emptyset$. Then $M \subset T(D(T) \cap G)$. If, moreover, T is of type (S) on ∂G , then $\overline{T(D(T) \cap \partial G)}$ can be replaced above by $T(D(T) \cap \partial G)$.*

Proof. We may assume that $0 \in D(T) \cap G$ and $0 \in T(0)$; otherwise, for some fixed $y_0 \in T(D(T) \cap G) \cap M$ and $x_0 \in D(T) \cap G$ such that $y_0 \in Tx_0$, we replace the operator T by the operator $\tilde{T}(x) \equiv T(x + x_0) - y_0$, $x \in D(\tilde{T})$, where $D(\tilde{T}) \equiv D(T) - x_0$. We also replace the sets M , G by the sets $\tilde{M} \equiv M - y_0$ and $\tilde{G} \equiv G - x_0$, respectively. It is easy to see that $0 \in D(\tilde{T}) \cap \tilde{G}$, $0 \in \tilde{T}(0)$, $0 \in \tilde{T}(D(\tilde{T}) \cap \tilde{G}) \cap \tilde{M}$, \tilde{T} is maximal monotone on $D(\tilde{T})$ and $\overline{\tilde{T}(D(\tilde{T}) \cap \partial \tilde{G})} \cap \tilde{M} = \emptyset$.

Let $y^* \in M$ be fixed. Since $0 \in M$ and M is pathwise connected, there exists a path $s : [0, 1] \rightarrow M$ such that $s(0) = 0$ and $s(1) = y^*$. Since T is maximal monotone, the operator $T_n \equiv T + (1/n)J$ is maximal monotone and surjective, for every $n = 1, 2, \dots$. Thus, for each n and $t \in [0, 1]$, the inclusion

$$(1) \quad Tx + (1/n)Jx \ni s(t)$$

is solvable. We show first that there exists a positive integer n_0 such that $T_n(D(T) \cap \partial G) \not\ni s(t)$, for any $n \geq n_0$ and any $t \in [0, 1]$. To this end, assume that the contrary is true and let $\{n_k\}$, a subsequence of $\{n\}$, and $t_k \in [0, 1]$, $k = 1, 2, \dots$, be such that $x_{n_k} \in D(T) \cap \partial G$ and

$$(2) \quad Tx_{n_k} + (1/n_k)Jx_{n_k} \ni s(t_k).$$

Since $[0, 1]$ is compact, we may also assume that $t_k \rightarrow t_0 \in [0, 1]$. Since G is bounded, there exists $y_{n_k}^* \in Tx_{n_k}$ such that $y_{n_k}^* \rightarrow s(t_0)$. It follows that $s(t_0) \in \overline{T(D(T) \cap \partial G)}$, i.e., a contradiction to our hypothesis on the set M . From this point on, we assume that $n \geq n_0$.

We now show that $y^* \in T_n(D(T) \cap G)$. The operator T_n has a continuous single-valued inverse $T_n^{-1} : X^* \rightarrow D(T)$. If we give $D(T)$ the relative topology induced on it by X , the mapping $T_n^{-1} : X^* \rightarrow D(T)$ is still continuous. Due to this important observation, its inverse image function takes a set open (closed) in $D(T)$ into a set which is open (closed) in X^* . Consequently, $T_n(D(T) \cap G)$ is open and $T_n(D(T) \cap \partial G)$ is closed in X^* . From

$$T_n(D(T) \cap \overline{G}) = T_n(D(T) \cap G) \cup T_n(D(T) \cap \partial G),$$

$$\overline{T_n(D(T) \cap G)} = T_n(D(T) \cap G) \cup \partial T_n(D(T) \cap G)$$

and

$$\overline{T_n(D(T) \cap G)} \subset \overline{T_n(D(T) \cap \overline{G})} = T_n(D(T) \cap \overline{G}),$$

it follows that

$$\partial T_n(D(T) \cap G) \subset T_n(D(T) \cap \partial G).$$

Consequently, since the path s does not have a point in common with the set $\partial T_n(D(T) \cap G)$, it must lie entirely in the set

$$T_n(D(T) \cap G) \cup \left(\overline{T_n(D(T) \cap G)}\right)^c,$$

where A^c denotes the complement of the set $A \subset X^*$. However, the sets

$$T_n(D(T) \cap G), \left(\overline{T_n(D(T) \cap G)}\right)^c$$

are separated. Since the range of s is connected, being the continuous image of $[0, 1]$, and $s(0) \in T_n(D(T) \cap G)$, we must have $s(t) \in T_n(D(T) \cap G)$, for every $t \in [0, 1]$. Thus, $y^* \in T_n(D(T) \cap G)$.

Let x_n denote the solution of

$$Tx + (1/n)Jx \ni y^*.$$

Since G is bounded, we may assume that, for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$, we have $x_{n_k} \rightarrow x \in \overline{coG}$. Then

$$\langle (1/n_i)Jx_{n_i} - (1/n_j)Jx_{n_j}, x_{n_i} - x_{n_j} \rangle = -\langle v_{n_i}^* - v_{n_j}^*, x_{n_i} - x_{n_j} \rangle \leq 0,$$

for all $i, j = 1, 2, \dots$, where $v_{n_i}^* \in Tx_{n_i}$, $v_{n_j}^* \in Tx_{n_j}$. Lemma A implies now that $x_{n_k} \rightarrow x \in \overline{G}$. Since $v_{n_k}^* \rightarrow y^*$ and T is maximal monotone, $x \in D(T)$ and $y^* \in Tx$. Since $M \cap T(D(T) \cap \partial G) = \emptyset$, $x \in D(T) \cap G$.

Now, let T be of type (S) on ∂G . Then, in the argument above about $s(t_0) \in \overline{T(D(T) \cap \partial G)}$, $x_{n_k} \rightarrow x_0$ and (2) imply

$$\langle y_{n_k}^*, x_{n_k} - x_0 \rangle = -\langle (1/n_k)x_{n_k}, x_{n_k} - x_0 \rangle + \langle s(t_k), x_{n_k} - x_0 \rangle,$$

for some $y_{n_k}^* \in Tx_{n_k}$, and

$$\lim_{k \rightarrow \infty} \langle y_{n_k}^*, x_{n_k} - x_0 \rangle = 0.$$

This implies that $x_{n_k} \rightarrow x_0 \in \partial G$, and the maximality of T yields $Tx_0 \ni s(t_0)$. Since $x_0 \in D(T) \cap \partial G$, we have a contradiction to our assumption on the set M . \square

Before we establish the invariance of domain result for maximal monotone operators, Proposition 1, we need the following corollary to Theorem 1.

Corollary 1. *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and let G be an open and bounded subset of X . Assume, further, that T is of type (S) on ∂G and*

$$T(D(T) \cap G) \cap T(D(T) \cap \partial G) = \emptyset.$$

Then $T(D(T) \cap G)$ is open.

Proof. Let $x_0 \in D(T) \cap G$ and $y_0 \in Tx_0$. As in the proof of Theorem 1, we may assume that $x_0 = 0$, $y_0 = 0$. We show that there exists $r > 0$ such that the open ball $B_r(0) \subset X^*$ lies in the set $T(D(T) \cap G)$.

We shall apply Theorem 1 with $M = B_r(0)$, for some $r > 0$. We show first that there exists $r > 0$ such that

$$(2) \quad T(D(T) \cap \partial G) \cap B_r(0) = \emptyset.$$

Let us assume that this is not true. Then there exists a sequence of positive numbers $\{r_n\}$ such that $r_n \rightarrow 0$ and

$$T(D(T) \cap \partial G) \cap B_{r_n}(0) \neq \emptyset.$$

This implies that there exists a sequence $\{x_n\} \subset D(T) \cap \partial G$ and a sequence $\{y_n^*\}$ such that $y_n^* \in Tx_n \cap B_{r_n}(0)$ and $y_n^* \rightarrow 0$ as $n \rightarrow \infty$. Since $\{x_n\}$ is bounded, we may assume that $x_n \rightarrow x_0 \in \text{co}G$. Then

$$\lim_{n \rightarrow \infty} \langle y_n^*, x_n - x_0 \rangle = 0.$$

Since T is of type (S) on ∂G , $x_n \rightarrow x_0 \in \partial G$ and, because T is maximal monotone, $0 \in Tx_0$ and $x_0 \in D(T) \cap \partial G$. This implies $0 \in T(D(T) \cap G) \cap T(D(T) \cap \partial G)$, i.e., a contradiction to our assumption. Thus, for some $r > 0$, (3) is true. Obviously, $B_r(0)$ is pathwise connected. Thus, we can apply Theorem 1 to conclude that $B_r(0) \subset T(D(T) \cap G)$. The proof is complete. \square

In the next statement we do not assume that G is bounded, but we do assume that T is locally injective and locally of type (S) . Given an operator $T : X \supset D(T) \rightarrow 2^{X^*}$, we say that T has a property \mathcal{P} “locally” on $G \subset X$ if for every $x_0 \in D(T) \cap G$ there exists a ball $B_r(x_0) \subset G$ such that T has the property \mathcal{P} on $D(T) \cap B_r(x_0)$. If $G = X$, then we say simply that T has “locally” the property \mathcal{P} .

Proposition 1. *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and let G be an open subset of X . Then the following statements are true.*

- (i) *If T is locally injective and of type (S) on G , then $T(D(T) \cap G)$ is open;*
- (ii) *if T is of type (S) , locally injective and $R(T)$ is closed in X^* , then $R(T) = X^*$.*

Proof. (i) Since T is locally injective and of type (S) on G , given $x_0 \in D(T) \cap G$, there exists a ball $\overline{B_q(x_0)} \subset G$ such that T is injective and of type (S) on $\overline{B_q(x_0)}$. Since $(D(T) \cap B_q(x_0)) \cap (D(T) \cap \partial B_q(x_0)) = \emptyset$, $T(D(T) \cap B_q(x_0)) \cap T(D(T) \cap \partial B_q(x_0)) = \emptyset$. By Corollary 1, the set $T(D(T) \cap B_q(x_0))$ is open. Since $T(D(T) \cap G)$ is a union of such sets, $T(D(T) \cap G)$ is also open.

(ii) By Part (i), for $G = X$, $R(T) = T(D(T) \cap X)$ is open. Since $R(T)$ is also closed, $R(T) = X^*$. \square

We now give a corollary to Theorem 1 which contains a criterion for the existence of a ball in the range of a maximal monotone operator T . For a set A we set $|A| \equiv \inf\{\|u\| : u \in A\}$.

Corollary 2. *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and let G be an open and bounded subset of X . If there exist $r > 0$, $x_0 \in D(T) \cap G$ such that*

$$|Tx_0| < r \leq |T(D(T) \cap \partial G)|,$$

then the ball $B_r(0) \subset X^$ lies in the range of the operator T . Actually, we have $T(D(T) \cap G) \supset B_r(0)$.*

Proof. Since

$$\left| \overline{T(D(T) \cap \partial G)} \right| \geq r,$$

we have

$$\overline{T(D(T) \cap \partial G)} \cap B_r(0) = \emptyset.$$

Also, since $|Tx_0| < r$, there exists $y_0^* \in Tx_0$ such that $\|y_0^*\| < r$. Thus, $y_0^* \in B_r(0)$ and

$$T(D(T) \cap G) \cap B_r(0) \neq \emptyset.$$

Consequently, we can apply Theorem 1 with $M \equiv B_r(0)$. The proof is complete. \square

Yang proved Corollary 2 in [18] by using Browder’s degree theory from [4].

We now give, for the first time, necessary and sufficient conditions for the existence of a zero of a completely continuous perturbation C of a maximal monotone operator T . This is a rather interesting result, which shows the power of the Leray-Schauder condition in the solvability of nonlinear inclusions involving some type of compactness or condensity. However, we do make the assumption that $T + C$ be locally monotone on an open and bounded set G .

Proposition 2. *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and $C : \overline{D(T)} \rightarrow X^*$ completely continuous. Assume, further, that $G \subset X$ is open and bounded and such that $T + C$ is locally monotone on G . Then the following conditions are equivalent.*

- (a) $0 \in (T + C)(D(T) \cap G)$;
- (b) *there exist $r > 0$ and $x_0 \in D(T) \cap G$ such that $\overline{B_r(x_0)} \subset G$ and $\langle u^* + Cx, x - x_0 \rangle \geq 0$, for every $(x, u^*) \in (D(T) \cap \partial B_r(x_0)) \times Tx$;*
- (c) *there exist $r > 0$ and $x_0 \in D(T) \cap G$ such that $\overline{B_r(x_0)} \subset G$ and*
 $(4) \quad (T + C)x \not\preceq \mu J(x - x_0)$, *for every $(\mu, x) \in (-\infty, 0) \times (D(T) \cap \partial B_r(x_0))$.*

Proof. Assume that $0 \in (T + C)(D(T) \cap G)$. Then there exists $x_0 \in D(T) \cap G$ such that $0 \in (T + C)x_0$. Since $T + C$ is locally monotone on G , there exists a ball $\overline{B_r(x_0)} \subset G$ such that $T + C$ is monotone on $D(T) \cap \overline{B_r(x_0)}$. Consequently,

$$\langle u^* + Cx, x - x_0 \rangle \geq 0, \text{ for every } (x, u^*) \in (D(T) \cap \partial B_r(x_0)) \times Tx;$$

It follows that (a) \Rightarrow (b).

To show that (b) \Rightarrow (c), assume that (b) holds and let $(T + C)x \ni \mu J(x - x_0)$, for some $(\mu, x) \in (-\infty, 0) \times (D(T) \cap \partial B_r(x_0))$. Then, for some $u^* \in Tx$,

$$0 \leq \langle u^* + Cx, x - x_0 \rangle = \mu \langle J(x - x_0), x - x_0 \rangle = \mu \|x - x_0\|^2 < 0.$$

This contradiction says that (b) \Rightarrow (c).

Let (c) hold. We note that we may assume that $x_0 = 0$ and $0 \in T(0)$; otherwise we replace Tx by $\tilde{T}x \equiv T(x + x_0) - v_0^*$ and Cx by $\tilde{C}x \equiv C(x + x_0) + v_0^*$, where $x \in D(\tilde{T}) \equiv D(T) - x_0$ and $v_0^* \in T(x_0)$. We also replace G by $\tilde{G} \equiv G - x_0$ and $B_r(x_0)$ by $B_r(0)$. Condition (4) would now read as follows:

$$(\tilde{T} + \tilde{C})x \not\preceq \mu Jx, \text{ for every } (\mu, x) \in (-\infty, 0) \times (D(\tilde{T}) \cap \partial B_r(0)).$$

We consider the approximate problem

$$(5) \quad Tx + Cx + (1/n)Jx \ni 0.$$

We observe that since the operator T is maximal monotone, the homotopy function

$$H(t, x) \equiv x - (tT + (1/n)J)^{-1}(-tCx), \quad (t, x) \in [0, 1] \times \overline{D(T)},$$

is well-defined. In order to define $H(t, \cdot)$ on the set $\overline{B_r(0)}$, we restrict the operator C on the closed and bounded set $\overline{D(T) \cap B_r(0)}$. Since X is reflexive, this restricted

operator C is compact. We extend C to a compact operator on all of X by Lemma 31 of Rothe [17]. We also use the symbol C to denote this new compact operator. Now, we observe that the mapping $G_t : x \rightarrow (tT + (1/n)J)^{-1}(-tCx)$ is compact for every $t \in [0, 1]$. Also, $G_t x$ is continuous in t uniformly w.r.t. $x \in \overline{B_r(0)}$ by the proofs of Theorems 3.2 and 3.7 of Guan and the author [7].

In order to be able to solve the equation $H(1, x) = 0$ (hence (5)), with $x \in B_r(0)$, using the Leray-Schauder theory, it suffices to show that $H(t, x) = 0$ has no solution $x_t \in \partial B_r(0)$, for any $t \in (0, 1]$. To this end, let $x_t \in \partial B_r(0)$ solve $H(t, x) = 0$, for some $t \in (0, 1]$. Then $x_t \in D(T)$ and

$$t(T + C)x_t + (1/n)Jx_t \ni 0,$$

or

$$Tx_t + Cx_t \ni -[1/(nt)]Jx_t,$$

which is a contradiction to (4). Let us call x_n a solution of (5) lying in $B_r(0)$. We may assume that

$$x_n \rightarrow \tilde{x} \in \overline{\text{co}(D(T) \cap B_r(0))} \subset \overline{\text{co}(\overline{D(T)} \cap \overline{B_r(0)})} = \overline{D(T)} \cap \overline{B_r(0)}.$$

Here, we have used the fact that $\overline{D(T)}$ is convex (cf. Zeidler [19, p. 915]). Since C is the original operator C on the set $\overline{D(T)} \cap \overline{B_r(0)}$, it is completely continuous there and $Cx_n \rightarrow C\tilde{x}$. Now, we may apply Property (b) in Zeidler [19, p. 915] to conclude that $\tilde{x} \in D(T)$ and $T\tilde{x} + C\tilde{x} \ni 0$. Since $\overline{B_r(0)} \subset G$, $\tilde{x} \in D(T) \cap G$. \square

We now consider another necessary and sufficient condition for the existence of a zero of a multi-valued maximal monotone operator T . Related conditions for demicontinuous monotone operators T can be found in the papers by Guan [6] and the author [13].

Theorem 2. *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and let G be an open, convex and bounded subset of X . Assume, further, that T is of type (S) on ∂G and*

$$T(D(T) \cap G) \cap T(D(T) \cap \partial G) = \emptyset.$$

Then the following are equivalent.

- (i) $0 \in T(D(T) \cap G)$;
- (ii) *there exists $x_0 \in D(T) \cap G$ such that $|Tx_0| \leq |Tx|$, for every $x \in D(T) \cap \partial G$.*

Proof. It is obvious that (i) implies (ii). To show that (ii) implies (i), we observe first that Corollary 1 implies that the set $T(D(T) \cap G)$ is open in X^* . To show that $T(D(T) \cap \overline{G})$ is closed, let $\{y_n^*\} \subset T(D(T) \cap \overline{G})$ be such that $y_n^* \rightarrow y_0^* \in X^*$. Then there exists a sequence $x_{n_k} \in D(T) \cap \overline{G}$ such that $x_{n_k} \rightarrow x_0 \in \overline{D(T)} \cap \overline{G}$ and $y_{n_k}^* \in Tx_{n_k}$. Invoking Problem 32.6 in Zeidler [19, p. 915], we have $x_0 \in D(T)$ and $y_0^* \in Tx_0$. Since $x_0 \in D(T) \cap \overline{G}$, $T(D(T) \cap \overline{G})$ is closed. Since $T(D(T) \cap G) \cap T(D(T) \cap \partial G) = \emptyset$, we use

$$\begin{aligned} T(D(T) \cap \overline{G}) &= T(D(T) \cap G) \cup T(D(T) \cap \partial G) \\ &\supset \overline{T(D(T) \cap G)} \\ &= T(D(T) \cap G) \cup \partial T(D(T) \cap G) \end{aligned}$$

to conclude that

$$\partial T(D(T) \cap G) \subset T(D(T) \cap \partial G).$$

We are going to show that the set

$$M = \cup\{tz^* : t \in [0, 1]\}$$

lies in the set $T(D(T) \cap G)$, where $z^* \in Tx_0$ is chosen so that $\|z^*\| = |Tx_0|$. This is possible due to the reflexivity and the local uniform convexity of the space X^* , and the fact that Tx_0 is a closed and convex set (cf. Zeidler [19, p. 915]). To this end, we show first that $M \cap T(D(T) \cap \partial G) = \emptyset$. Let $y^* \in M \cap T(D(T) \cap \partial G)$. Then $y^* = tz^*$, for some $t \in [0, 1]$, and $y^* \in T\bar{x}$, for some $\bar{x} \in D(T) \cap \partial G$. If $z^* = 0$, we are done. Thus, we may assume that $z^* \neq 0$. If $y^* = 0$, then $\|z^*\| \leq |T\bar{x}| \leq \|y^*\| = 0$, i.e., a contradiction. It follows that $y^* \neq 0$, which says that there exists $t \in (0, 1]$ such that $y^* = tz^*$. If $t = 1$, then $y^* \in T(D(T) \cap G) \cap T(D(T) \cap \partial G)$, i.e., a contradiction. Thus, $t \in (0, 1)$ and $t\|z^*\| < |Tx_0| \leq |T\bar{x}| \leq \|y^*\| = t\|z^*\|$. This contradiction proves our assertion. Since the set M is connected, $M \cap T(D(T) \cap G) \ni z^*$ and $M \cap \partial T(D(T) \cap G) = \emptyset$, we must have $M \subset T(D(T) \cap G)$. The proof is complete. \square

3. DISCUSSION

It does not seem possible to have an exact analogue of the proof Theorem 1 for m -accretive operators T , even in the case of uniformly convex X and X^* , unless the duality mapping is weakly continuous in a neighborhood of zero and T satisfies a certain type of closedness condition. Nevertheless, versions of Corollary 2 and a part of Proposition 2 exist for such operators in various papers in the references and some of the references therein. For example, the author has shown in [12] the following relevant result. The space X is now a general Banach space.

Theorem A. *Let $T : X \supset D(T) \rightarrow 2^X$ be m -accretive and $C : X \supset \bar{G} \rightarrow X$ compact, where G is open and bounded. Assume that there exists $x_0 \in D(T) \cap G$ such that*

$$Tx + Cx \not\supseteq \mu(x - x_0), \quad \text{for every } (\mu, x) \in (-\infty, 0) \times (\partial G \cap D(T)).$$

Then $0 \in \overline{(T+C)(D(T) \cap G)}$. Moreover, $0 \in (T+C)(D(T) \cap G)$ under one of the following conditions:

- (i) X is uniformly convex, G is convex and C is completely continuous;
- (ii) T is strongly accretive on $D(T) \cap G$;
- (iii) the compactness condition on C is replaced by the following: C is bounded, continuous and the resolvent $(T + I)^{-1}$ is compact.

Thus, a version of the sufficiency part of Part (iii) of Proposition 2 holds for m -accretive operators without the local accretivity condition on the set G . Actually, a theorem like Theorem A above holds for maximal monotone operators. We give this version below for the sake of completeness. We should also mention that necessary and sufficient conditions for the existence of a zero of a maximal monotone operator have been given by the author in [13], but the proofs there are considerably different from the ones herein. We assume below the usual properties of the spaces X , X^* as in Section 2.

Theorem 3. Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and $C : X \supset \overline{G} \rightarrow X^*$ compact, where G is open and bounded. Assume that there exists $x_0 \in D(T) \cap G$ such that

$$Tx + Cx \not\equiv \mu J(x - x_0), \quad \text{for every } (\mu, x) \in (-\infty, 0) \times (\partial G \cap D(T)).$$

Then $0 \in \overline{(T + C)(D(T) \cap G)}$. Moreover, $0 \in (T + C)(D(T) \cap G)$ under one of the following conditions:

- (i) G is convex and C is completely continuous;
- (ii) T is strongly monotone on $D(T) \cap G$;
- (iii) the compactness condition on C is replaced by the following: C is bounded, continuous and the resolvent $(T + J)^{-1}$ is compact.

Proof. The proof is mainly a combination of the proof of Proposition 2 and the proof of Proposition 1 of [12]. It is therefore omitted. \square

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