

## ON THE BOCHNER–RIESZ MEANS OF CRITICAL ORDER

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ABSTRACT. Stein’s well-known logarithmic asymptotics of the Lebesgue constants of the Bochner-Riesz means of critical order is extended to Lebesgue constants of more general linear means of multiple Fourier series. These means are generated by certain class of functions supported in convex domains with boundaries of non-vanishing Gaussian curvature.

### 1. INTRODUCTION

Let  $f$  be an integrable function on  $\mathbb{T}^n = (-\pi, \pi]^n$ ,  $2\pi$ -periodic in each variable. Let  $f$  have a multiple Fourier series

$$\sum_k \hat{f}(k) e^{ik \cdot x},$$

where  $x = (x_1, \dots, x_n)$  is a point in the real  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ ,  $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$  – the lattice of points in  $\mathbb{R}^n$  with integer coordinates,  $k \cdot x = k_1 x_1 + \dots + k_n x_n$  is the scalar product, and

$$\hat{f}(k) = (2\pi)^{-n} \int_{\mathbb{T}^n} f(x) e^{-ik \cdot x} dx$$

is the  $k$ -th Fourier coefficient of function  $f$ .

Consider a sequence of linear operators

$$L_N^\lambda : f \mapsto L_N^\lambda(f; x) = \sum_k \lambda\left(\frac{k}{N}\right) \hat{f}(k) e^{ik \cdot x},$$

where  $\lambda$  is a continuous function. The operators map  $C(\mathbb{T}^n)$  into  $C(\mathbb{T}^n)$ , or  $L_1(\mathbb{T}^n)$  into  $L_1(\mathbb{T}^n)$ . It is well known that the norms in both cases are the same. They are called the Lebesgue constants. When  $\lambda$  has a compact support, say  $S$ , the norms can be written explicitly

$$\|L_N^\lambda\| = (2\pi)^{-n} \int_{\mathbb{T}^n} \left| \sum_k \lambda\left(\frac{k}{N}\right) e^{ik \cdot x} \right| dx.$$

The summation is taken over all  $k \in NS \cap \mathbb{Z}^n$ , where  $NS = \{y : y = Nx, x \in S\}$ .

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The most famous case is represented by the Bochner-Riesz means ( $\lambda(x) = R_\alpha(x) = (1 - |x|^2)_+^\alpha$ ,  $\alpha \geq 0$ ), see [Bc]. The value  $\frac{n-1}{2}$  is called critical. Stein's well-known result [S] says that

$$(1) \quad \left\| L_N^{R_{\frac{n-1}{2}}} \right\| = \omega_n \ln N + o(\ln N),$$

where  $\omega_n$  is some positive constant depending only on  $n$  (not indicated explicitly in [S]). This asymptotics makes it clear why it is often said that the Bochner-Riesz means of critical order behave as one-dimensional partial sums. In [Be] and [BL] this result was obtained as a simple corollary of certain (different) rather general theorems.

Our main goal is to generalize this result to a wider class of functions  $\lambda$  (in connection with their supports). Let  $S$  be the compact support of a function  $\lambda$ , with the boundary  $\partial S$ . Let  $\lambda(x) = \lambda_{p,\alpha}(x)$  be  $p$ -smooth inside  $S$ , and in a certain neighborhood of  $\partial S$  may be represented as follows:

$$\lambda(x) = \lambda_{p,\alpha}(x) = f(x)(\rho(x))^\alpha,$$

where  $f \in C^p(\mathbb{R}^n)$  and does not vanish on  $\partial S$ ,  $\rho(x) = 0$  if  $x \notin S$ ,  $\rho(x) = \text{dist}(x, \partial S)$  if  $x \in S$ . Notice, that  $\rho$  is a smooth function in a neighborhood of  $\partial S$  (see e.g., [Gi, Appendix B]).

Throughout the text we denote by  $C, C_1$  some absolute positive constants, possibly different even if the same notation appears at one line.

## 2. BASIC RESULTS

**Theorem.** *Let  $S$  be the compact support of a function  $\lambda = \lambda_{n, \frac{n-1}{2}}$ , with the  $n$ -smooth boundary  $\partial S$ . Assume that  $S$  is convex and the principal curvatures of  $\partial S$  never vanish. Then there exists a positive constant  $C_{S,\lambda}$  depending only on  $S$  and  $\lambda$  such that*

$$\|L_N^\lambda\| = C_{S,\lambda} \ln N + o(\ln N)$$

for large  $N$ .

The proof relies on two basic facts. The following theorem, which is a part of Belinskii's result [Be, Theorem 1], gives the first basic fact.

Let  $\Delta_z^m(\lambda; h_1, \dots, h_m)$  be the  $m$ -th difference of a function  $\lambda$  defined recursively by the formulas

$$\begin{aligned} \Delta_z^1(\lambda; h_1) &= \lambda(z + h_1) - \lambda(z); \\ \Delta_z^m(\lambda; h_1, \dots, h_m) &= \Delta_{z+h_m}^{m-1}(\lambda; h_1, \dots, h_{m-1}) - \Delta_z^{m-1}(\lambda; h_1, \dots, h_{m-1}), \end{aligned}$$

where  $h_j, z \in \mathbb{R}^n$ ,  $m$  is an arbitrary integer. Let

$$\hat{\lambda}(x) = \int_{\mathbb{R}^n} \lambda(u) e^{iux} du$$

be the Fourier transform of any integrable function  $\lambda$ .

**Theorem A.** For each boundedly supported continuous function  $\lambda$

$$(2) \quad \|L_N^\lambda\| = (2\pi)^{-n} \int_{N\mathbb{T}^n} \left| \prod_{j=1}^n \frac{x_j}{2N \sin \frac{x_j}{2N}} \hat{\lambda}(x) \right| dx + O \left\{ \sum_{j=1}^{m-1} \int_{N\mathbb{T}^n} \left| \frac{x}{N} \right|^j |\hat{\lambda}(x)| dx \right. \\ \left. + \int_{\frac{1}{2\pi}\mathbb{T}^n} \cdots \int_{\frac{1}{2\pi}\mathbb{T}^n} \left[ \sum_k \left| \Delta_{\frac{k}{N}}^m \left( \lambda; \frac{u_1}{N}, \dots, \frac{u_m}{N} \right) \right|^2 \right]^{\frac{1}{2}} du_1 \dots du_m \right\}.$$

The proof of this result may be found also in Appendix to the paper [LRZ].

In order to apply Theorem A, we need some information about the behavior of the Fourier transform. The asymptotics of the Fourier transform of functions of the type specified is our second basic fact. This result is a consequence of a theorem describing the singularities of the Radon transform of such functions (see [RZ1], [RZ2]) and may be found in [LRZ] (see also [RZ3]).

**Theorem B.** Let  $S$  be the compact support of a function  $\lambda = \lambda_{p,\alpha}$  with  $\alpha \geq 0$  and  $p > \max(1, \frac{n-1}{2} + \alpha)$ . Let  $S$  be convex, with the  $p$ -smooth boundary  $\partial S$ , and suppose the principal curvatures of  $\partial S$  never vanish. Let  $\theta \in \mathbb{R}^n$  be a vector on the unit sphere,  $x^+(\theta)$  and  $x^-(\theta)$  be the (uniquely defined) points of  $\partial S$  at which the function  $\theta_1 x_1 + \cdots + \theta_n x_n$  attains maximum and minimum on  $\partial S$ , respectively. Then for  $t \rightarrow +\infty$

$$\hat{\lambda}(t\theta_1, \dots, t\theta_n) = t^{-\alpha - \frac{n+1}{2}} \left( \Xi^+ e^{it\theta x^+(\theta)} + \Xi^- e^{it\theta x^-(\theta)} + o(1) \right), \\ \Xi^\pm = (2\pi)^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) \exp\left\{ \pm \pi i \frac{2\alpha + n + 1}{4} \right\} f(x^\pm) (\varkappa^\pm)^{-\frac{1}{2}},$$

where the remainder term is small uniformly in  $\theta$ , and  $\varkappa^\pm$  is the Gaussian curvature of  $\partial S$  at the points  $x^\pm$ , respectively.

### 3. PROOF OF THEOREM

Let us use (2) with  $m$  specified below, and obtain the asymptotics of the first term in the right-hand side of (2), and all the rest are estimated from above. Let us start with the estimates of the remainder terms in (2). As for the integrals from the sum in the first term of the remainder in (2), using spherical coordinates and applying Theorem B, we get

$$(3) \quad \int_0^{CN} N^{-j} t^j t^{n-1} t^{-\frac{n+1}{2} - \frac{n-1}{2}} dt \leq C_1 N^{-j} N^{\frac{n-1}{2} - \frac{n-1}{2} + j} = O(1)$$

for each  $j = 1, \dots, m-1$ . Now consider the last term in (2). Our computations are similar to those used by Belinskii for the usual Bochner-Riesz means [Be] and to those fulfilled in general situation in [LRZ] (see also [CoS]). It suffices to estimate

the larger quantity

$$\begin{aligned}
 & \sup_{u_1, \dots, u_m \in \frac{1}{2\pi} \mathbb{T}^n} \left\{ \sum_k \left| \Delta_{\frac{k}{N}}^m \left( \lambda; \frac{u_1}{N}, \dots, \frac{u_m}{N} \right) \right|^2 \right\}^{\frac{1}{2}} \\
 &= \sup_{u_1, \dots, u_m \in \frac{1}{2\pi} \mathbb{T}^n} \left\{ \sum_{m+2 \leq \text{dist}(k, \partial(NS)) \leq CN} \left| \Delta_{\frac{k}{N}}^m \left( \lambda; \frac{u_1}{N}, \dots, \frac{u_m}{N} \right) \right|^2 \right. \\
 (4) \quad & \left. + \sum_{0 \leq \text{dist}(k, \partial(NS)) \leq m+2} \left| \Delta_{\frac{k}{N}}^m \left( \lambda; \frac{u_1}{N}, \dots, \frac{u_m}{N} \right) \right|^2 \right\}^{\frac{1}{2}}.
 \end{aligned}$$

We may estimate from above each term in the last sum by the maximal value of  $\lambda$  at the points with integer coordinates. This yields

$$\sum_{0 \leq \text{dist}(k, \partial(NS)) \leq m+2} \left| \Delta_{\frac{k}{N}}^m \left( \lambda; \frac{u_1}{N}, \dots, \frac{u_m}{N} \right) \right|^2 \leq C \sum_{0 \leq \text{dist}(k, \partial(NS)) \leq m+2} \left( \frac{m+2}{N} \right)^{2 \frac{n-1}{2}}.$$

The number of points with integer coordinates inside  $NY$ , where  $Y \subset \mathbb{R}^n$  is a convex domain, is equal to  $N^n \text{vol}(Y) + O(N^{n-1})$ , see e.g., [V]. Therefore, the last sum is not greater than

$$(5) \quad CN^{-2 \frac{n-1}{2}} \sum_{0 \leq \text{dist}(k, \partial(NS)) \leq m+2} 1 \leq C_1 N^{n-1-2 \frac{n-1}{2}} = O(1).$$

Now consider the first sum on the right-hand side of (4). Use  $m$  times the mean value theorem for the directional derivative. The sum being estimated is transformed to the form

$$N^{-2m} \sum_{m+2 \leq \text{dist}(k, \partial(NS)) \leq CN} \left| \frac{\partial^m \lambda \left( \frac{k}{N} + \sigma_1 \frac{u_1}{N} + \dots + \sigma_m \frac{u_m}{N} \right)}{\partial u_1 \dots \partial u_m} \right|^2.$$

Here  $0 < \sigma_j < 1, j = 1, \dots, m$ . We can take any  $m$  such that  $\frac{n+1}{2} \leq m \leq n$ . If  $\frac{n-1}{2}$  is an integer the derivative is bounded, so this quantity is not greater than  $CN^{n-2-2 \frac{n-1}{2}} = O(1)$ . Otherwise, we estimate the derivative in the direction  $(\sigma_1, \dots, \sigma_m)$  by its maximal value, and obtain

$$\begin{aligned}
 & N^{-2m} \sum_{m+2 \leq \text{dist}(k, \partial(NS)) \leq CN} \left| \frac{\partial^m \lambda \left( \frac{k}{N} + \sigma_1 \frac{u_1}{N} + \dots + \sigma_m \frac{u_m}{N} \right)}{\partial u_1 \dots \partial u_m} \right|^2 \leq \\
 (6) \quad & C_1 N^{-2m} \sum_{m+2 \leq \text{dist}(k, \partial(NS)) \leq CN} \left[ \text{dist} \left( \frac{k + \sigma_{m,k}}{N}, \partial S \right) \right]^{2 \left( \frac{n-1}{2} - m \right)},
 \end{aligned}$$

where  $0 \leq \sigma_{m,k} \leq m$ . Let us denote  $k + \sigma_{m,k}$  by  $\bar{k}$ . We have provided  $m > \frac{n-1}{2}$ :

$$\begin{aligned}
 & N^{-2m} \sum_{\frac{m+2}{N} \leq \text{dist}(\frac{k}{N}, \partial S) \leq C} \left[ \text{dist} \left( \frac{\bar{k}}{N}, \partial S \right) \right]^{2(\frac{n-1}{2}-m)} \\
 &= N^{-2m} \sum_{m+2 \leq q \leq CN} \sum_{\frac{q}{N} \leq \text{dist}(\frac{k}{N}, \partial S) \leq \frac{q+1}{N}} \left[ \text{dist} \left( \frac{\bar{k}}{N}, \partial S \right) \right]^{n-1-2m} \\
 (7) \quad & \leq CN^{-2m} \sum_{1 \leq q \leq CN} \sum_{\frac{q}{N} \leq \text{dist}(\frac{k}{N}, \partial S) \leq \frac{q+1}{N}} \left( \frac{q}{N} \right)^{n-1-2m} \\
 &= CN^{-n+1} \sum_{1 \leq q \leq CN} q^{n-1-2m} \sum_{\frac{q}{N} \leq \text{dist}(\frac{k}{N}, \partial S) \leq \frac{q+1}{N}} 1 \\
 & \leq CN^{-n+1} N^{n-1} \sum_{1 \leq q \leq CN} q^{n-1-2m} \leq C(1 + N^{n-1-2m+1}) \\
 & \leq C(1 + N^{n-2\frac{n+1}{2}}) = O(1).
 \end{aligned}$$

Collecting (3)–(7), we obtain that the remainder in (2) is bounded. Now we go on to the estimate of the main term. Let us pass to the spherical coordinates and apply Theorem B. Taking into account also that

$$\frac{x_j}{2N \sin \frac{x_j}{2N}} - 1 = O\left(\frac{x_j^2}{N^2}\right)$$

which leads to estimates like the ones for the remainder terms (see (6)), we get the following in place of the main term:

$$(2\pi)^{-n} (2\pi)^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{|\theta|=1} d\theta \int_1^{\pi N} \left| \Xi^+(\theta) e^{itx^+(\theta)\theta} + \Xi^-(\theta) e^{itx^-(\theta)\theta} \right| \frac{dt}{t} + o(\ln N).$$

Consider the inner integral in an equivalent form

$$\int_1^{\pi N} \left| \Xi^+(\theta) e^{it(x^+(\theta)-x^-(\theta))\theta} + \Xi^-(\theta) \right| \frac{dt}{t},$$

and estimate it, denoting  $(x^+(\theta) - x^-(\theta))\theta$  by  $d(\theta)$ . The following relation is very well-known (see e.g., [Z, Vol.1, Ch.2]):

$$(8) \quad \int_1^{\pi N} \left| \frac{\sin t}{t} \right| dt = \frac{2}{\pi} \ln N + O(1).$$

Let us adopt the method used in [Z] to get (8) to our conditions. The integral under estimation is equal to

$$\int_{d(\theta)}^{\pi N d(\theta)} \left| \Xi^+(\theta) e^{it} + \Xi^-(\theta) \right| \frac{dt}{t} = \sum_{k=1}^{[Nd(\theta)]} \int_{2k\pi}^{2(k+1)\pi} \left| \Xi^+(\theta) e^{it} + \Xi^-(\theta) \right| \frac{dt}{t} + O(1).$$

The integral on the right-hand side is equal to

$$\begin{aligned} & \sum_{k=1}^{[Nd(\theta)]} \int_{\frac{2k\pi}{Nd(\theta)}}^{\frac{2(k+1)\pi}{Nd(\theta)}} \left| \Xi^+(\theta)e^{itNd(\theta)} + \Xi^-(\theta) \right| \frac{dt}{t} \\ &= \int_0^{\frac{2\pi}{Nd(\theta)}} \left| \Xi^+(\theta)e^{itNd(\theta)} + \Xi^-(\theta) \right| \sum_{k=1}^{[Nd(\theta)]} \frac{1}{t + \frac{2k\pi}{Nd(\theta)}} dt. \end{aligned}$$

The last sum is equal to  $\frac{Nd(\theta)}{2\pi} [\ln Nd(\theta) + O(1)]$ . Thus, we need to estimate

$$\begin{aligned} & \frac{1}{2\pi} (2\pi)^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{|\theta|=1} Nd(\theta) [\ln Nd(\theta) + O(1)] d\theta \\ & \quad \cdot \int_0^{\frac{2\pi}{Nd(\theta)}} \left| \Xi^+(\theta)e^{itNd(\theta)} + \Xi^-(\theta) \right| dt \\ &= (2\pi)^{-\frac{n+3}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{|\theta|=1} [\ln Nd(\theta) + O(1)] d\theta \int_0^{2\pi} \left| \Xi^+(\theta)e^{it} + \Xi^-(\theta) \right| dt \\ &= \ln N (2\pi)^{-\frac{n+3}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{|\theta|=1} d\theta \int_0^{2\pi} \left| \Xi^+(\theta)e^{it} + \Xi^-(\theta) \right| dt + O(1). \end{aligned}$$

Denoting  $f(x^\pm(\theta))(\varkappa^\pm(\theta))^{-\frac{1}{2}}$  by  $\phi^\pm(\theta)$  (it should be recalled that  $f$  and the Gaussian curvature  $\varkappa$  do not vanish anywhere) we have to estimate

$$\begin{aligned} & (2\pi)^{-\frac{n+3}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{|\theta|=1} d\theta \int_0^{2\pi} \left| \phi^+(\theta)e^{\frac{i\pi n}{2}} e^{it} + \phi^-(\theta)e^{-\frac{i\pi n}{2}} \right| dt \\ (9) \quad &= (2\pi)^{-\frac{n+3}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{|\theta|=1} d\theta \int_0^{2\pi} \left| (-1)^n \phi^+(\theta)e^{it} + \phi^-(\theta) \right| dt. \end{aligned}$$

But the expression in the inner integral on the right-hand side of (9) never vanishes, and the right-hand side of (9) may be denoted by  $C_{S,\lambda}$ . The proof is complete.

*Remark 1.* Let us find  $C_{S,\lambda}$  for usual Bochner-Riesz means of critical order; that is, let us find  $\omega_n$  in (1). Here  $S$  is the unit sphere, so  $\varkappa = 1$  everywhere. Since  $\lambda(x) = (1 - |x|^2)_+^{\frac{n-1}{2}}$ , the function  $f(x) = (1 + |x|)^{\frac{n-1}{2}}$ . On the boundary, that is, for  $|x| = 1$ , we get  $f(x^\pm(\theta)) = 2^{\frac{n-1}{2}}$ . Taking into account the following well-known identity

$$\int_{|\theta|=1} d\theta = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

we get

$$\begin{aligned}\omega_n &= (2\pi)^{-n+3} \Gamma\left(\frac{n+1}{2}\right) \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} 2^{\frac{n-1}{2}} \int_0^{2\pi} |(-1)^n e^{it} + 1| dt \\ &= \frac{1}{2} \pi^{-\frac{3}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} 8 = \frac{4\Gamma\left(\frac{n+1}{2}\right)}{\pi^{3/2} \Gamma\left(\frac{n}{2}\right)}.\end{aligned}$$

The same value was obtained in [BL] and, after correcting a small misprint, in [Be].

*Remark 2.* Generalizations of the Bochner-Riesz means of order  $\alpha$  smaller than  $\frac{n-1}{2}$  were considered earlier. Upper estimates of the Lebesgue constants were obtained in [CoS], and lower estimates in [LRZ], respectively. As in the spherical case (see [Ba], [IA]) the degree of growth of the Lebesgue constants is  $N^{\frac{n-1}{2}-\alpha}$ . For  $\alpha > \frac{n-1}{2}$ , the corresponding Lebesgue constants are bounded (see [CoS], it is also obvious in our case).

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