CANONICAL SYSTEMS
AND TRANSFER MATRIX-FUNCTIONS

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Abstract. The generalised Backlund-Darboux transform and its modifications are applied to the inverse spectral problems and problems on similarity of Hamiltonians for canonical systems.

1. Introduction

The spectral theory of canonical systems

\[ w(x, \lambda) = \lambda JH(x)w(x, \lambda), \quad H = H^* \in L_{m \times m}^1(0, l), \quad J = -J^* = -J^{-1}, \]
\[ w(0, \lambda) = E_m = \{ \delta_{kj} \}_{k,j=1}^m; \]

\[ J = \{ J_{kj} \}_{k,j=1}^2, \quad J_{11} = J_{22} = 0, \quad J_{12} = -J_{21} = -E_p; \quad m = 2p; \quad H \geq 0 \]

has both mathematical and physical significance and was investigated in many brilliant works (see [1],[2] and references therein). The method of the operator identities [3] proved to be fruitful in the spectral theory of systems (1), (2). It dealt with a set of operators

\[ A, S \in \{ L_m^2(0, l), L_m^2(0, l) \}, \quad \Pi = [\Phi_1, \Phi_2] \in \{ C^m, L_m^2(0, l) \}. \]

(We say \( K \in \{ \Xi_1, \Xi_2 \} \) if \( K \) is a bounded operator acting from Hilbert space \( \Xi_1 \) into Hilbert space \( \Xi_2 \). \( C \) is the complex plane.) The operators \( A, S, \Pi \) are supposed to satisfy the identity

\[ AS - SA^* = \Pi J \Pi^*. \]

Representation of \( w(x, \lambda) \) in the form of the transfer matrix-function

\[ w(l, \lambda) = E_m + \lambda J \Pi^* S^{-1}(E - \lambda A)^{-1}\Pi \]

and representation of the operators \( S, \Phi_1 \) through the fixed operators \( A, \Phi_2 \) and spectral function \( \tau \) allowed one to solve a wide class of inverse spectral problems (problems of restoration \( H \) by \( \tau \)). Backlund and Darboux have found connections between different solutions of the sine-Gordon equation \( u_{xt} = \sin u \) so that, starting from one solution \( u_0 \), a whole family of solutions could be constructed. Later it
was discovered (see [4]) that these connections have a far more general character and are based on the spectral properties of certain transformations of the solution and coefficient $W$ and $\xi$ of the system

$$W_x = (i\lambda_j + \xi(x))W, \quad W(0, \lambda) = E_m, \quad \xi^*j = -j\xi, \quad j = \{(−1)^{k+1}\delta_{kr}E_p\}_{k,r=1}^2.$$  

This system is equivalent to (1), where $H = TW^*(x, 0)W(x, 0)T^*$, $JH$ is similar to $J$ ($T = (1/\sqrt{2})\{T_{kr}\}_{k,r=1}^2$, $T_{12} = -T_{11} = E_p$, $T_{21} = T_{22} = iE_p$). The spectral properties of the Backlund-Darboux transform (BDT) were considered in the well-known papers by Flashke, McLaughlin and Walquist (see [4]). The method of the operator identities was used in the further investigations of BDT [5]–[7]. Here we apply generalised BDT (with $H$ not necessarily similar to $J$) to obtain solutions of the inverse problems for canonical systems. In our case the internal operator $A$ is not fixed (as in [3]) but depends on spectral data. We consider mostly the case $\det H(x) \neq 0$, when spectral functions of the canonical system coincide with the spectral functions of the minimal operator generated by the expression $-H^{-1}(x)J(d/dx)$.

2. BDT AND TRANSFER MATRIX-FUNCTION

Let an arbitrary system (1) ($H(x) = H^*(x)$) and the $n \times n$ matrix $A$ be given. Then an $n \times m$ matrix-function $\Pi(x)$ shall be introduced by the equality

$$\Pi_x = -AIJH \quad (\Pi_x = d\Pi/dx).$$

(When $A$ is diagonal, the columns of $\Pi^*$ are eigenvectors of system (1).) If $\sigma(A) \cap \sigma(A^*) = \emptyset$ (\(\sigma\)-spectrum), then the operator identity

$$AS - SA^* = \Pi J\Pi^*$$

has a unique solution $S(x)$. By (3), (4) the $n \times n$ matrix-function $S(x)$ satisfies the equality

$$S_x = \Pi JHJ^*\Pi^*.$$

In the general case we introduce $S(x)$ by the equation (5) and the initial value $S(0)$, that satisfies the conditions

$$AS(0) - S(0)A^* = \Pi(0)J\Pi^*(0), \quad S(0) = S^*(0).$$

(The existence of such an $S(0)$ is demanded. The identity (4) then follows from (5), (6).) According to the general results on the BDT and transfer matrix-functions [7] the transfer matrix-function

$$w_A(x, \lambda) = E_m - J\Pi^*(x)S^{-1}(x)(A - \lambda E_n)^{-1}\Pi(x)$$

satisfies the equation

$$\begin{equation}
(w_A(x, \lambda))_x = (\lambda JH(x) - \tilde{q}_0(x))w_A(x, \lambda) - \lambda w_A(x, \lambda)JH(x),
\end{equation}$$

where $\tilde{q}_0 = JH\Pi^*S^{-1}\Pi - J\Pi^*S^{-1}\Pi JH$. (This fact easily follows from (3)–(5).)

Putting

$$\begin{align*}
\tilde{w}(x, \lambda) &= v(x, \lambda)w(x, \lambda)v^{-1}(0, \lambda), \\
v(x, \lambda) &= w_0^{-1}(x)w_A(x, \lambda), \\
w_0(0) &= E_m,
\end{align*}$$

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and taking into account (1), (8), we obtain the result
\[ w(x, \lambda) = \lambda J \tilde{H}(x) \hat{w}(x, \lambda), \quad \hat{H}(x) = w_0(x) H(x) w_0(x). \]

**Theorem 1.** Let the matrix-functions \( w, \Pi \) satisfy (1), (3). Then equalities (5)–(7) define a transfer matrix-function \( w_A \), such that \( \tilde{w} \) of the form (9) satisfies system (10).

**Remark 2.** Equalities (9), (10) give us the iterated BDT of the \( w \) and \( H \) of system (1). When \( JH \) is similar to \( J, n = 1 \) our BDT coincides with the classical BDT.

### 3. Canonical systems. Weyl functions

We shall demand here that conditions (2) and \( S(0) > 0 \) hold. Then, taking into account (5), we get \( S(x) > 0 \). By virtue of (4), (7) it is true [3], that \( w_A^*(x, \lambda)Jw_A(x, \mu) = J + (\mu - \lambda)\Pi^*(x)(A^* - \bar{X}E_n)^{-1}S^{-1}(x)(A - \mu E_n)^{-1}\Pi(x) \). Hence \( w_A^*(x, \lambda)Jw_A(x, \lambda) = J, \ iJ^*w_A^*(x, \lambda)Jw_A(x, \lambda)J \leq iJ \) \((\text{Im} \lambda > 0) \). According to definition (9) of \( v^{-1} \) and the equality \( w_0^*Jw_0 = J \), we obtain now
\[ iv^{-1}(x, \lambda)J[v^*(x, \lambda)]^{-1} \leq iJ \quad (\text{Im} \lambda > 0). \]

**Definition 3.** A pair of \( p \times p \) matrix-functions \( P(\lambda), Q(\lambda) \), meromorphic in the upper half-plane is called nonsingular with \( J \)-property, if
\[ i[P^*(\lambda) Q^*(\lambda)]J \begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix} \leq 0, \quad P^*(\lambda)P(\lambda) + Q^*(\lambda)Q(\lambda) > 0. \]

By virtue of (11), if the pair \( P, Q \) satisfies (12), then \( \tilde{P}, \tilde{Q} \) of the form
\[ \begin{bmatrix} \tilde{P} \\ \tilde{Q} \end{bmatrix} = [v^*(l, \lambda)]^{-1} \begin{bmatrix} P \\ Q \end{bmatrix} \]
satisfy (12) also. Let us consider now the canonical system (1), (2) on the interval \((0, l)\), supposing for simplicity \( \text{det} H(x) \neq 0 \) almost everywhere. By \( N(\Omega) \) we shall designate the set of functions of the form
\[ \phi(\lambda) = [\Omega_{11}(\lambda)P(\lambda) + \Omega_{12}(\lambda)Q(\lambda)][\Omega_{21}(\lambda)P(\lambda) + \Omega_{22}(\lambda)Q(\lambda)]^{-1}, \]
where \( P, Q \) satisfy (12), \( \Omega(\lambda) = [\Omega_{kj}(\lambda)]_{k,j=1}^2 = w^*(l, \lambda) \). It is easily seen, that \( \text{Im} \phi(\lambda) = \text{Im} \phi^*(\lambda) - \phi(\lambda) \geq 0 \) \((\text{Im} \lambda \geq 0)\), i.e. \( \phi(\lambda) \in N(\Omega) \) are Nevanlinna functions. So every \( \phi(\lambda) \) defines a unique distribution function \( \tau(t) \) in the representation
\[ \phi(\lambda) = a\lambda + \beta + \int_{-\infty}^{\infty} ((t - \lambda)^{-1} - t(1 + t^2)^{-1})d\tau(t) \quad (a \geq 0, \ \beta = \beta^*). \]

According to [8] (see also the references therein) the set \( N(H) \) of spectral functions of systems (1), (2) coincides with the set of distribution functions for \( \phi(\lambda) \in N(\Omega) \). The functions \( \phi(\lambda) \in N(\Omega) \) are called Weyl functions of systems (1), (2). The connections between the spectral functions of systems (1), (2) and (10) will be investigated in terms of \( N(\Omega) \) and \( N(\tilde{\Omega}) \) \((\tilde{\Omega} = \tilde{w}^*(l, \lambda) \).

By (9), (13), (14) the matrix-function \( \tilde{\phi}(\lambda) = [\tilde{\Omega}_{11}(\lambda)\tilde{P}(\lambda) + \tilde{\Omega}_{12}(\lambda)\tilde{Q}(\lambda)][\tilde{\Omega}_{21}(\lambda)\tilde{P}(\lambda) + \tilde{\Omega}_{22}(\lambda)\tilde{Q}(\lambda)]^{-1} \) can be presented in the form
\[ \tilde{\phi}(\lambda) = [\nu_{11}(\lambda)\phi(\lambda) + \nu_{12}(\lambda)][\nu_{21}(\lambda)\phi(\lambda) + \nu_{22}(\lambda)]^{-1}, \]
where \( \nu(\lambda) = \{\nu_{kj}(\lambda)\}_{k,j=1}^2 = [v^*(0, \lambda)]^{-1} \). As \( \tilde{P}, \tilde{Q} \) satisfies (12), so \( \tilde{\phi}(\lambda) \in N(\tilde{\Omega}) \).
Theorem 4. If \( H > 0 \), \( S(0) > 0 \), \( \phi(\lambda) \in N(\Omega) \), then it is true, that \( \tilde{\phi}(\lambda) \) of the form (16) belongs to \( N(\tilde{\Omega}) \), i.e. \( \tilde{\phi} \) is a Weyl function of system (10). Formula (16) gives us the transformation of the Weyl function due to the iterated BDT of system (1), (2).

4. Canonical systems. Transformations of the point spectrum of spectral matrix-functions

Let us consider now the case

\[
A = \text{Diag}\{a_1, \ldots\} = A^*, \quad a_k \neq a_j \quad (k \neq j);
\]

\[
\Pi = [\Phi_1 \Phi_2], \quad \Phi_2(0) = 0; \quad S(0) = E_n.
\]

From (9) it follows that \( \nu(\lambda) = E_m - \Pi(0)S^{-1}(0)(A - \lambda E_n)^{-1}\Pi(0)J \). Hence \( \nu_{11} = \nu_{22} = E_P, \nu_{21} = 0 \). By virtue of (16) we get \( \tilde{\phi}(\lambda) = \phi(\lambda) + \nu_{12}(\lambda) = \phi(\lambda) + \sum_{k=1}^{n} \Phi_{1k}^*(0)\Phi_{1k}(0)/(\lambda_k - \lambda) \), where \( \Phi_{1k} \) are rows of \( \Phi_1 \).

Theorem 5. Let conditions (17) hold. Then the distribution matrix-function \( \tilde{\tau} \) for \( \tilde{\phi}(\lambda) \) is defined through \( \tau \) and the piecewise constant matrix-function \( \tau_0 \) with jumps at the points \( \{a_k\} \):

\[
\tilde{\tau}(t) = \tau(t) + \tau_0(t), \quad \tau_0(a_k + 0) - \tau_0(a_k - 0) = \Phi_{1k}^*(0)\Phi_{1k}(0).
\]

Corollary 6. Let the distribution matrix-function \( \tilde{\tau}(t) \) of the form (18) and Hamiltonian \( H(x) \) of system (1), (2) with spectral function \( \tau(t) \) be given. Then \( \tilde{\tau}(t) \) is a spectral function of the canonical system with Hamiltonian \( \tilde{H} = w_0^*Hw_0 \), where

\[
(w_0)_x = -\tilde{q}_0w_0, \quad w_0(0) = E_m, \quad \tilde{q}_0 = JH\Pi^*S^{-1}\Pi - \Pi^*S^{-1}\Pi JH.
\]

5. Inverse problem. General case

Now let the nondecreasing \( p \times p \) matrix-function \( \tau_0(t) \) satisfy condition

\[
\sup_{x \leq t} \left\| \int_{-\infty}^{x} w(x,t) \begin{bmatrix} 0 \\ E_p \end{bmatrix} d\tau_0(t)[0 E_p]w^*(x,t) \right\| < \infty.
\]

Then the operators \( \Pi(x) \):

\[
\Pi f = [0 E_p]w^*(x,t)Jf
\]

belong to the set \( \{C^m, L^2(\tau_0)\} \) of bounded operators, acting from \( C^m \) to \( L^2(\tau_0) \). Introducing operators \( A \), acting in \( L^2(\tau_0) \), \( S(\lambda) \in \{ L^2(\tau_0), L^2(\tau_0) \} \), by the equalities

\[
A\psi = t\psi(t), \quad S\psi = \psi + [0 E_p] \int_{-\infty}^{x} s(x,t,u) \begin{bmatrix} 0 \\ E_p \end{bmatrix} d\tau_0(u)\psi(u),
\]

\[
s(x,t,u) = (w^*(x,t)Jw(x,u) - J)/(t - u)
\]

we obtain the operator identity \( AS - SA^* = \Pi J\Pi^* \). (The equality \( s(x,t,u) = \int_{0}^{x} w^*(r,t)H(r)w(r,u) dr \) is essential to show that \( S\psi, \psi \) is defined for \( \psi \) with finite support, \( S \) is bounded and \( S \geq E \).) According to (1), (21), (22), similarly to (3) we have \( (\Pi f)_x = -\Pi JH(x)f \) (for \( \tau_0 \) with finite support). Therefore analogously to previous considerations, we prove that equation (8) is still valid. Thus the generalization of Corollary 6 is true, when (20) is true (case of infinite support for \( \tau_0 \) included).
Theorem 7. Let the distribution matrix-function $\tilde{\tau} = \tau + \tau_0$ and the Hamiltonian $\tilde{H}(x) > 0$ of system (1), (2) with spectral matrix-function $\tau$ be given. Suppose function $\tau_0$ is nondecreasing and satisfies (20). Then $\tilde{\tau}$ is a spectral function of the canonical system with Hamiltonian $\tilde{H} = w_0^*Hw_0$, where $w_0$ is defined by (19).

The problem of obtaining general conditions, under which solutions $JH$ and $J\tilde{H}$ of the inverse spectral problems are similar, was stated in [3] (and is closely connected with the investigation of the classes of Hamiltonians, for which the inverse problem has a unique solution). Theorem 7 gives sufficient conditions of similarity.

References


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