

CANONICAL SYSTEMS AND TRANSFER MATRIX-FUNCTIONS

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ABSTRACT. The generalised Backlund-Darboux transform and its modifications are applied to the inverse spectral problems and problems on similarity of Hamiltonians for canonical systems.

1. INTRODUCTION

The spectral theory of canonical systems

$$(1) \quad \begin{aligned} w_x(x, \lambda) &= \lambda JH(x)w(x, \lambda), \quad H = H^* \in L^1_{m \times m}(0, l), J = -J^* = -J^{-1} \\ (w(0, \lambda) &= E_m = \{\delta_{kj}\}_{k,j=1}^m); \end{aligned}$$

$$(2) \quad J = \{J_{kj}\}_{k,j=1}^2, J_{11} = J_{22} = 0, J_{12} = -J_{21} = -E_p; \quad m = 2p; \quad H \geq 0$$

has both mathematical and physical significance and was investigated in many brilliant works (see [1],[2] and references therein). The method of the operator identities [3] proved to be fruitful in the spectral theory of systems (1), (2). It dealt with a set of operators

$$A, S \in \{L^2_m(0, l), L^2_m(0, l)\}, \quad \Pi = [\Phi_1 \ \Phi_2] \in \{C^m, L^2_m(0, l)\}.$$

(We say $K \in \{\Xi_1, \Xi_2\}$ if K is a bounded operator acting from Hilbert space Ξ_1 into Hilbert space Ξ_2 . C is the complex plane.) The operators A, S, Π are supposed to satisfy the identity

$$AS - SA^* = \Pi J \Pi^*.$$

Representation of $w(x, \lambda)$ in the form of the transfer matrix-function

$$w(l, \lambda) = E_m + \lambda J \Pi^* S^{-1} (E - \lambda A)^{-1} \Pi$$

and representation of the operators S, Φ_1 through the fixed operators A, Φ_2 and spectral function τ allowed one to solve a wide class of inverse spectral problems (problems of restoration H by τ). Backlund and Darboux have found connections between different solutions of the sine-Gordon equation $u_{xt} = \sin u$ so that, starting from one solution u_0 , a whole family of solutions could be constructed. Later it

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was discovered (see [4]) that these connections have a far more general character and are based on the spectral properties of certain transformations of the solution and coefficient W and ξ of the system

$$W_x = (i\lambda j + \xi(x))W, \quad W(0, \lambda) = E_m, \quad \xi^* j = -j\xi, \quad j = \{(-1)^{k+1} \delta_{kr} E_p\}_{k,r=1}^2.$$

This system is equivalent to (1), where $H = TW^*(x, 0)W(x, 0)T^*$, JH is similar to j ($T = (1/\sqrt{2})\{T_{kr}\}_{k,r=1}^2$, $T_{12} = -T_{11} = E_p$, $T_{21} = T_{22} = iE_p$). The spectral properties of the Backlund-Darboux transform (BDT) were considered in the well-known papers by Flashke, McLaughlin and Walquist (see [4]). The method of the operator identities was used in the further investigations of BDT [5]–[7]. Here we apply generalised BDT (with H not necessarily similar to j) to obtain solutions of the inverse problems for canonical systems. In our case the internal operator A is not fixed (as in [3]) but depends on spectral data. We consider mostly the case $\det H(x) \neq 0$, when spectral functions of the canonical system coincide with the spectral functions of the minimal operator generated by the expression $-H^{-1}(x)J(d/dx)$.

2. BDT AND TRANSFER MATRIX-FUNCTION

Let an arbitrary system (1) ($H(x) = H^*(x)$) and the $n \times n$ matrix A be given. Then an $n \times m$ matrix-function $\Pi(x)$ shall be introduced by the equality

$$(3) \quad \Pi_x = -A\Pi JH \quad (\Pi_x = d\Pi/dx).$$

(When A is diagonal, the columns of Π^* are eigenvectors of system (1).) If $\sigma(A) \cap \sigma(A^*) = \emptyset$ (σ -spectrum), then the operator identity

$$(4) \quad AS - SA^* = \Pi J\Pi^*$$

has a unique solution $S(x)$. By (3), (4) the $n \times n$ matrix-function $S(x)$ satisfies the equality

$$(5) \quad S_x = \Pi JH J^* \Pi^*.$$

In the general case we introduce $S(x)$ by the equation (5) and the initial value $S(0)$, that satisfies the conditions

$$(6) \quad AS(0) - S(0)A^* = \Pi(0)J\Pi^*(0), \quad S(0) = S^*(0).$$

(The existence of such an $S(0)$ is demanded. The identity (4) then follows from (5), (6).) According to the general results on the BDT and transfer matrix-functions [7] the transfer matrix-function

$$(7) \quad w_A(x, \lambda) = E_m - J\Pi^*(x)S^{-1}(x)(A - \lambda E_n)^{-1}\Pi(x)$$

satisfies the equation

$$(8) \quad (w_A(x, \lambda))_x = (\lambda JH(x) - \tilde{q}_0(x))w_A(x, \lambda) - \lambda w_A(x, \lambda)JH(x),$$

where $\tilde{q}_0 = JH J\Pi^* S^{-1}\Pi - J\Pi^* S^{-1}\Pi JH$. (This fact easily follows from (3)–(5).) Putting

$$(9) \quad \begin{aligned} \tilde{w}(x, \lambda) &= v(x, \lambda)w(x, \lambda)v^{-1}(0, \lambda), \\ v(x, \lambda) &= w_0^{-1}(x)w_A(x, \lambda), \\ (w_0)_x &= -\tilde{q}_0 w_0, \quad w_0(0) = E_m, \end{aligned}$$

and taking into account (1), (8), $w_0^* J w_0 = J$, we obtain the result

$$(10) \quad \tilde{w}_x(x, \lambda) = \lambda J \tilde{H}(x) \tilde{w}(x, \lambda), \quad \tilde{H}(x) = w_0^*(x) H(x) w_0(x).$$

Theorem 1. *Let the matrix-functions w, Π satisfy (1), (3). Then equalities (5)–(7) define a transfer matrix-function w_A , such that \tilde{w} of the form (9) satisfies system (10).*

Remark 2. Equalities (9), (10) give us the iterated BDT of the w and H of system (1). When JH is similar to j , $n = 1$ our BDT coincides with the classical BDT.

3. CANONICAL SYSTEMS. WEYL FUNCTIONS

We shall demand here that conditions (2) and $S(0) > 0$ hold. Then, taking into account (5), we get $S(x) > 0$. By virtue of (4), (7) it is true [3], that $w_A^*(x, \lambda) J w_A(x, \mu) = J + (\mu - \bar{\lambda}) \Pi^*(x) (A^* - \bar{\lambda} E_n)^{-1} S^{-1}(x) (A - \mu E_n)^{-1} \Pi(x)$. Hence $w_A^*(x, \lambda) J w_A(x, \bar{\lambda}) = J, iJ^* w_A^*(x, \lambda) J w_A(x, \lambda) J \leq iJ$ ($\text{Im } \lambda > 0$). According to definition (9) of v^{-1} and the equality $w_0^* J w_0 = J$, we obtain now

$$(11) \quad i v^{-1}(x, \bar{\lambda}) J [v^*(x, \bar{\lambda})]^{-1} \leq iJ \quad (\text{Im } \lambda > 0).$$

Definition 3. A pair of $p \times p$ matrix-functions $P(\lambda), Q(\lambda)$, meromorphic in the upper half-plane is called nonsingular with J -property, if

$$(12) \quad i [P^*(\lambda) \ Q^*(\lambda)] J \begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix} \leq 0, \quad P^*(\lambda) P(\lambda) + Q^*(\lambda) Q(\lambda) > 0.$$

By virtue of (11), if the pair P, Q satisfies (12), then \tilde{P}, \tilde{Q} of the form

$$(13) \quad \begin{bmatrix} \tilde{P} \\ \tilde{Q} \end{bmatrix} = [v^*(l, \bar{\lambda})]^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}$$

satisfy (12) also. Let us consider now the canonical system (1), (2) on the interval $(0, l)$, supposing for simplicity $\det H(x) \neq 0$ almost everywhere. By $N(\Omega)$ we shall designate the set of functions of the form

$$(14) \quad \phi(\lambda) = [\Omega_{11}(\lambda) P(\lambda) + \Omega_{12}(\lambda) Q(\lambda)] [\Omega_{21}(\lambda) P(\lambda) + \Omega_{22}(\lambda) Q(\lambda)]^{-1},$$

where P, Q satisfy (12), $\Omega(\lambda) = \{\Omega_{kj}(\lambda)\}_{k,j=1}^2 = w^*(l, \bar{\lambda})$. It is easily seen, that $\text{Im } \phi(\lambda) = i(\phi^*(\lambda) - \phi(\lambda)) \geq 0$ ($\text{Im } \lambda \geq 0$), i.e. $\phi(\lambda) \in N(\Omega)$ are Nevanlinna functions. So every $\phi(\lambda)$ defines a unique distribution function $\tau(t)$ in the representation

$$(15) \quad \phi(\lambda) = \alpha \lambda + \beta + \int_{-\infty}^{\infty} ((t - \lambda)^{-1} - t(1 + t^2)^{-1}) d\tau(t) \quad (\alpha \geq 0, \beta = \beta^*).$$

According to [8] (see also the references therein) the set $N(H)$ of spectral functions of systems (1), (2) coincides with the set of distribution functions for $\phi(\lambda) \in N(\Omega)$. The functions $\phi(\lambda) \in N(\Omega)$ are called Weyl functions of systems (1), (2). The connections between the spectral functions of systems (1), (2) and (10) will be investigated in terms of $N(\Omega)$ and $N(\tilde{\Omega})$ ($\tilde{\Omega} = \tilde{w}^*(l, \bar{\lambda})$). By (9), (13), (14) the matrix-function $\tilde{\phi}(\lambda) = [\tilde{\Omega}_{11}(\lambda) \tilde{P}(\lambda) + \tilde{\Omega}_{12}(\lambda) \tilde{Q}(\lambda)] [\tilde{\Omega}_{21}(\lambda) \tilde{P}(\lambda) + \tilde{\Omega}_{22}(\lambda) \tilde{Q}(\lambda)]^{-1}$ can be presented in the form

$$(16) \quad \tilde{\phi}(\lambda) = [\nu_{11}(\lambda) \phi(\lambda) + \nu_{12}(\lambda)] [\nu_{21}(\lambda) \phi(\lambda) + \nu_{22}(\lambda)]^{-1},$$

where $\nu(\lambda) = \{\nu_{kj}(\lambda)\}_{k,j=1}^2 = [v^*(0, \bar{\lambda})]^{-1}$. As \tilde{P}, \tilde{Q} satisfies (12), so $\tilde{\phi}(\lambda) \in N(\tilde{\Omega})$.

Theorem 4. *If $H > 0$, $S(0) > 0$, $\phi(\lambda) \in N(\Omega)$, then it is true, that $\tilde{\phi}(\lambda)$ of the form (16) belongs to $N(\tilde{\Omega})$, i.e. $\tilde{\phi}$ is a Weyl function of system (10). Formula (16) gives us the transformation of the Weyl function due to the iterated BDT of system (1), (2).*

4. CANONICAL SYSTEMS. TRANSFORMATIONS OF THE POINT SPECTRUM OF SPECTRAL MATRIX-FUNCTIONS

Let us consider now the case

$$(17) \quad \begin{aligned} A &= \text{Diag}\{a_1, \dots\} = A^*, & a_k &\neq a_j \quad (k \neq j); \\ \Pi &= [\Phi_1 \ \Phi_2], & \Phi_2(0) &= 0; & S(0) &= E_n. \end{aligned}$$

From (9) it follows that $\nu(\lambda) = E_m - \Pi^*(0)S^{-1}(0)(A - \lambda E_n)^{-1}\Pi(0)J$. Hence $\nu_{11} = \nu_{22} = E_p$, $\nu_{21} = 0$. By virtue of (16) we get $\tilde{\phi}(\lambda) = \phi(\lambda) + \nu_{12}(\lambda) = \phi(\lambda) + \sum_{k=1}^n \Phi_{1k}^*(0)\Phi_{1k}(0)/(a_k - \lambda)$, where Φ_{1k} are rows of Φ_1 .

Theorem 5. *Let conditions (17) hold. Then the distribution matrix-function $\tilde{\tau}$ for $\tilde{\phi}(\lambda)$ is defined through τ and the piecewise constant matrix-function τ_0 with jumps at the points $\{a_k\}$:*

$$(18) \quad \tilde{\tau}(t) = \tau(t) + \tau_0(t), \quad \tau_0(a_k + 0) - \tau_0(a_k - 0) = \Phi_{1k}^*(0)\Phi_{1k}(0).$$

Corollary 6. *Let the distribution matrix-function $\tilde{\tau}(t)$ of the form (18) and Hamiltonian $H(x)$ of system (1), (2) with spectral function $\tau(t)$ be given. Then $\tilde{\tau}(t)$ is a spectral function of the canonical system with Hamiltonian $\tilde{H} = w_0^*Hw_0$, where*

$$(19) \quad (w_0)_x = -\tilde{q}_0 w_0, \quad w_0(0) = E_m, \quad \tilde{q}_0 = JHJ\Pi^*S^{-1}\Pi - J\Pi^*S^{-1}\Pi JH.$$

5. INVERSE PROBLEM. GENERAL CASE

Now let the nondecreasing $p \times p$ matrix-function $\tau_0(t)$ satisfy condition

$$(20) \quad \sup_{x \leq l} \left\| \int_{-\infty}^{\infty} w(x, t) \begin{bmatrix} 0 \\ E_p \end{bmatrix} d\tau_0(t) [0 \ E_p] w^*(x, t) \right\| < \infty.$$

Then the operators $\Pi(x)$:

$$(21) \quad \Pi f = [0 \ E_p] w^*(x, t) J f$$

belong to the set $\{C^m, L^2(\tau_0)\}$ of bounded operators, acting from C^m to $L^2(\tau_0)$. Introducing operators A , acting in $L^2(\tau_0)$, $S(x) \in \{L^2(\tau_0), L^2(\tau_0)\}$, by the equalities

$$(22) \quad \begin{aligned} A\psi &= t\psi(t), & S\psi &= \psi + [0 \ E_p] \int_{-\infty}^{\infty} s(x, t, u) \begin{bmatrix} 0 \\ E_p \end{bmatrix} d\tau_0(u)\psi(u), \\ s(x, t, u) &= (w^*(x, t)Jw(x, u) - J)/(t - u) \end{aligned}$$

we obtain the operator identity $AS - SA^* = \Pi J \Pi^*$. (The equality $s(x, t, u) = \int_0^x w^*(r, t)H(r)w(r, u) dr$ is essential to show that $(S\psi, \psi)$ is defined for ψ with finite support, S is bounded and $S \geq E$.) According to (1), (21), (22), similarly to (3) we have $(\Pi f)_x = -t\Pi J H(x)f$ (for τ_0 with finite support). Therefore analogously to previous considerations, we prove that equation (8) is still valid. Thus the generalization of Corollary 6 is true, when (20) is true (case of infinite support for τ_0 included).

Theorem 7. *Let the distribution matrix-function $\tilde{\tau} = \tau + \tau_0$ and the Hamiltonian $H(x) > 0$ of system (1), (2) with spectral matrix-function τ be given. Suppose function τ_0 is nondecreasing and satisfies (20). Then $\tilde{\tau}$ is a spectral function of the canonical system with Hamiltonian $\tilde{H} = w_0^* H w_0$, where w_0 is defined by (19).*

The problem of obtaining general conditions, under which solutions JH and $J\tilde{H}$ of the inverse spectral problems are similar, was stated in [3] (and is closely connected with the investigation of the classes of Hamiltonians, for which the inverse problem has a unique solution). Theorem 7 gives sufficient conditions of similarity.

REFERENCES

1. I. Gohberg and M. G. Krein, *Theory and applications of Volterra operators in Hilbert space*, Translated in: Transl. of Math. Monographs **24**, Providence, Rhode Island, 1970. MR **41**:9041
2. L. de Branges, *Hilbert spaces of entire functions*, N.J., Prentice-Hall, 1968. MR **37**:4590
3. L. A. Sakhnovich, *Factorisation problems and operator identities*, Uspekhi Mat. Nauk **41** (1986), no. 1, 3–55. Translated in: Russian Math. Surv. **41** (1986).
4. R. Miura (ed.), *Backlund transformations*, Lecture Notes in Math. **515**, Springer, 1976 MR **58**:15770
5. A. L. Sakhnovich, *Exact solutions of nonlinear equations and the method of operator identities*, Lin. Alg. Appl. **182** (1993), 109–126. MR **94e**:35044
6. A. L. Sakhnovich, *Dressing procedure for solutions of nonlinear equations and the method of operator identities*, Inverse Pr. **10** (1994), 699–710. MR **95k**:35175
7. A. L. Sakhnovich, *Iterated Darboux transform (the case of rational dependence on spectral parameter)*, Dokl. Akad. Nauk Ukraine (1995), no. 7, 24–27. CMP 96:06
8. A. L. Sakhnovich, *Spectral functions of a canonical system of order $2n$* . Translated in: Math. USSR Sbornik **71** (1992), no. 2, 355–369.

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