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A SIMPLE PROOF OF P. CARTER'S THEOREM

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Abstract. We give a simple proof of the following result of P. Carter: Given a twist homeomorphism of an annulus with at most one fixed point in the interior of the annulus, then there exists an essential simple closed curve inside this annulus meeting its image in at most the (possible) interior fixed point.

0. Introduction

Let us first recall the classical

Theorem (Poincaré-Birkhoff)**.** *Let F be a twist homeomorphism of the annulus $A = S^1 \times [0,1]$. If F preserves the area then F has at least two fixed points.*

Very soon, Birkhoff, Kérékjártó and others looked for a more topological statement avoiding the area preserving hypothesis. They obtained the following [B], [K]:

Theorem. *Let F be a twist homeomorphism of the annulus without fixed points in $intA$. Then there is an essential simple closed curve $c \subset intA$ such that $F(c) \cap c = \emptyset$.*

Of course this statement implies that in the classical setting one gets one fixed point. A natural question now is: is it possible to get the second fixed point working along the same lines? It was not before 1982 that the answer came out with the following result of P. Carter [C]:

Theorem. *Let F be a twist homeomorphism of the annulus A with at most one fixed point in $intA$. Then there is an essential simple closed curve $c \subset intA$ which meets its image in at most one point (the fixed point of F in $intA$, if it exists).*

In this note we propose to give a simplified proof of this theorem. In fact we shall prove a slightly stronger result. But let us first recall the

Definition. A homeomorphism F, isotopic to the identity, of the annulus $A = S^1 \times [0,1]$ is a **twist homeomorphism** if there is a lift f of F or F^{-1} to the band $B = \mathbb{R} \times [0,1]$ satisfying $f_1(x,0) < x$ and $f_1(x,1) > x$, where $f = (f_1, f_2)$. (We think of S^1 as $\mathbb{R}/\mathbb{Z}$).

Then the theorem to be proved is

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Theorem. *Let F a twist homeomorphism of the annulus A with a lift f of F (or F^{-1}) to the band B as in the previous definition, and suppose that f has at most $\mathbb{Z}\times\{\frac{1}{2}\}$ as set of fixed points (that is, either f has no fixed point or the fixed point set of f is $\mathbb{Z}\times\{\frac{1}{2}\}$). Then there is an essential simple closed curve $c\subset intA$ which meets its image in at most one point (which is the image of $\mathbb{Z}\times\{\frac{1}{2}\}$ by the projection map in case this is the fixed point set of f).*

Notice that F is allowed to have a large fixed point set (possibly with nonempty interior) in A.

It is known that one can rapidly deduce the theorem of Birkhoff and Kérékjártó from the Brouwer plane translation theorem [K], [G], and we shall follow the same path, using the recent proof of that theorem given by P. Le Calvez and A. Sauzet [LS] (but as we shall need to start from scratch, in view of the fixed point, this paper is almost self contained). Other proofs of Brouwer's theorem known to me do not seem to lead to such an easy proof, as the Brouwer lines they construct may converge on a fixed point (compare with lemma 2 below).

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1. Preliminaries

To prove the theorem it is of course enough to consider the case where f is a lift of F (rather than one of F^{-1}), and we can also suppose that f has $\mathbb{Z}\times\{\frac{1}{2}\}$ as set of fixed points (in which case F fixes $\{1\}\times\{\frac{1}{2}\}$): if f has no fixed point then the result follows from the proof of the Birkhoff-Kérékjártó theorem alluded to above and in any case can be obtained as a simplification of what follows.

We first remark that F admits a lift $\tilde{F}$, still covered by f, to a high enough cover $\tilde{A}=\mathbb{R}/n\mathbb{Z}\times[0,1]$ (n large) whose fixed point set is exactly

$$\{(0,\tfrac{1}{2}),(1,\tfrac{1}{2}),\dots,((n-1),\tfrac{1}{2})\}.$$

Up to the end of the proof we shall fix our attention on the homeomorphism $\tilde{F}$ and show that there exists a simple essential closed curve c in int$\tilde{A}$ such that $\tilde{F}(c)$ (or $\tilde{F}^{-1}(c)$) is contained in int$c\cup$ Fix$\tilde{F}$; then a final "descent argument" will conclude the proof.

Notice that all n fixed points of $\tilde{F}$ have the same index, which is 0 according to the Lefschetz formula.

Now we extend $\tilde{F}$ to a homeomorphism of $\mathbb{R}/n\mathbb{Z}\times[-\delta,1]$ in such a way that it admits a lift $f:\mathbb{R}\times[-\delta,1]\to\mathbb{R}\times[-\delta,1]$ such that

i)	$f(x,t)=(f_1(x,t),t)$	if $-\delta\le t\le 0$,
ii)	$f_1(x,t)<x$	if $-\delta\le t\le 0$,
iii)	$f(x,-\delta)=(x-\alpha,-\delta)$	for some small $\alpha>0$,
iv)	$f_1(x,t)>x$	if $t=1$.

Convention. All int, Fr, Adh or $\overline{}$ below are with respect to $\mathbb{R}^2$.

Definition. 1. A **brick decomposition** of a subset E in some surface is a collection $\{V_i\}_{i\in\mathbb{N}}$ of closed discs such that

i) $\bigcup_{i=0}^{\infty}V_i=E$.

ii) Every point of E admits a neighborhood which meets at most three of the V_i's.

iii) If $V_i\cap V_j\neq\emptyset$ then $V_i\cap V_j$ is a (nondegenerate) arc in Fr$V_i\cap$FrV_j.

2. A brick decomposition is said to be **generic** (with respect to some homeomorphism f of E) if every arc γ in the family $\{V_i \cap V_j\}_{i,j\in\mathbb{N}}$ satisfies: for every arc $\gamma' = V_k \cap V_l$ in the same family such that $f(\gamma) \cap \gamma' \neq \emptyset$ (resp. $f^{-1}(\gamma) \cap \gamma' \neq \emptyset$), the set $f(\gamma)$ (resp.$f^{-1}(\gamma)$) meets both intV_k and intV_l.

Note that the union of the elements of any finite subcollection of the V_i's is a 2-submanifold of E with boundary.

Definition. A subset X of some space endowed with a homeomorphism f is **free** if $f(X) \cap X = \emptyset$.

Lemma 1. *There exists a generic brick decomposition of* $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$, $\{V_i\}_{i\in\mathbb{N}}$ *which satisfies:*

(1) It is n-periodic (i.e. $\tau(V_j) \in \{V_i\}_{i\in\mathbb{N}}$ *for all* $V_j \in \{V_i\}_{i\in\mathbb{N}}$, *where* $\tau(x,t) = (x+n,t)$*).*

(2) Each V_i *is free.*

(3)

$$V_0 \cap \mathbb{R} \times \{-\delta\} = [0, \frac{2}{3}\alpha] \times \{-\delta\},$$

$$V_2 \cap \mathbb{R} \times \{-\delta\} = [\frac{2}{3}\alpha, \frac{4}{3}\alpha] \times \{-\delta\},$$

$$V_1 \cap \mathbb{R} \times \{-\delta\} = [-\frac{2}{3}\alpha, 0] \times \{-\delta\}$$

(therefore $f(V_0) \cap V_1 \neq \emptyset \neq f^{-1}(V_0) \cap V_2$*).*

Proof. We shall construct a brick decomposition$\{W_i\}$ of $\mathbb{R}/n\mathbb{Z} \times [-\delta, 1] \setminus \mathrm{Fix}\tilde{F}$ which will then be lifted to $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$. We first construct the bricks along $S^1 \times \{-\delta\}$ so that they are free under F and satisfy the obvious analog of (3), which is certainly possible if α is small enough, and we complete this decomposition to a free brick decomposition covering $\mathbb{R}/n\mathbb{Z} \times [-\delta, 1] \setminus \mathrm{int}B_3$ where B_k is the union of the n closed balls of radius $\frac{1}{k}$ ($k \geq 3$) centered at the fixed points of $\tilde{F}$. We then complete the brick decomposition successively on each $B_{k+1} \setminus \mathrm{int}B_k$ by bricks free under $\tilde{F}$. The lift of the decomposition so obtained satisfies the lemma except perhaps for genericity. To get genericity, choose some numbering $\{\gamma_k\}_{k\in\mathbb{N}}$ of the set $\{W_i \cap W_j\}_{i,j\in\mathbb{N}}$ and modify γ_0 slightly (if necessary) so that it becomes generic. Then modify γ_1 so slightly that γ_0 is still generic and γ_1 becomes generic. We continue in this way; each γ_i is modified only a finite number of times and we get a brick decomposition $\{W_i\}_{i\in\mathbb{N}}$ whose lift to $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$ is generic and satisfies (1), (2), (3) if our perturbations have been small enough. □

Lemma 2. *Let* $\{V_i\}_{i\in\mathbb{N}}$ *be a brick decomposition of* $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$ *and for* $X \subset \mathbb{N}$, *let* $W_X = int(\bigcup_{i\in X} V_i)$. *Then if* W_X *is connected, unbounded and if* $\mathbb{R}^2 \setminus \overline{W}_X$ *has no bounded components,* $\overline{W}_X$ *is a submanifold whose boundary contains no bounded components (i.e. circles).*

Proof. $\overline{W}_X$ is certainly a 2-manifold away from $\mathrm{Fix}f = \mathbb{Z} \times \{\frac{1}{2}\}$. Let z be a fixed point in $\mathrm{Fr}\overline{W}_X$ and let C be a small circle around z. Near C, $\overline{W}_X$ is a manifold and we can suppose C transversal to $\mathrm{Fr}\overline{W}_X$ so that $C \cap \overline{W}_X$ is some finite number of arcs and $C \cap \mathrm{Fr}\overline{W}_X$ an even number of points. Some of these points are paired by an arc in $\mathrm{Fr}\overline{W}_X \setminus \{z\}$ to another one while the other points of $C \cap \mathrm{Fr}\overline{W}_X$ are

joined to z by an arc of $\mathrm{Fr}\overline{W}_X$. Also $\mathrm{int}C$ does not contain any circle in $\mathrm{Fr}\overline{W}_X$ since $\mathbb{R}^2 \setminus \overline{W}_X$ has no bounded components. Since $\overline{W}_X$ is connected and $\mathbb{R}^2 \setminus \overline{W}_X$ has no bounded component, only two arcs from $C \cap \mathrm{Fr}\overline{W}_X$ to z can exist and $\overline{W}_X$ is a 2-submanifold of $\mathbb{R}^2$. The assertion on the boundary now follows as a circle in $\partial\overline{W}_X$ would have to bound a disc included in $\overline{W}_X$ or $\mathbb{R}^2 \setminus \overline{W}_X$. □

We will also have to use the following result of Franks [F, Proposition 1.3], which follows easily from Brouwer's lemma that an orientation preserving homeomorphism of the plane with a periodic point must have some compact fixed point set of positive index. To get the statement below from Franks' proposition, note that every fixed point of f has index 0 as this is the case for $\tilde{F}$ as noticed above.

Lemma 3. *There is no finite family $D_0, \dots, D_n = D_0$ of free disjoint open discs in $\mathbb{R} \times [-\delta, 1]$ such that $f(D_i) \cap D_{i+1} \neq \emptyset,\ 0 \leq i < n$.*

2. Proof of the theorem

Let us consider a brick decomposition $\{V_i\}_{i\in\mathbb{N}}$ of $\mathbb{R} \times [-\delta, 1] \setminus \mathbb{Z} \times \{\frac{1}{2}\}$ as given by Lemma 1.

Let $W_1 = \mathrm{int}(\bigcup\{V_i \big| \ \mathrm{int}V_i \cap f(\mathrm{int}V_0) \neq \emptyset\})$ and $W_n = \mathrm{int}(\bigcup\{V_i \big| \ \mathrm{int}V_i \cap f(W_{n-1}) \neq \emptyset\})$, $n > 1$.

Then $W_+ = \bigcup_{n\geq 1} W_n$ is a connected unbounded set. It is unbounded because by (3) of lemma 1, $\overline{W}_+ \supset]-\infty, 0[\ \times\{-\delta\}$. To prove it is connected we first remark that W_n is connected, since the open set $f(W_{n-1})$ is clearly covered by some set of V_i's so that $W_n \supset f(W_{n-1})$. Secondly we note that $V_1 \subset W_1$ by (3), so that if V_k meets $f(V_1 \cap V_0)$ then by transversality $V_k \subset W_1 \cap W_2$ and so $W_1 \cap W_2 \neq \emptyset$; therefore $f(W_1) \cap f(W_2) \neq \emptyset$, and by the same reasoning $W_2 \cap W_3 \neq \emptyset$ and more generally $W_n \cap W_{n+1} \neq \emptyset$, $n \geq 1$.

By construction (and genericity) $W_{n+1} \supset f(\overline{W}_n)$, so that $f(W_+) \subset W_+$, and if $x \in \mathrm{Fr}W_+$ then $f(x) \in W_+$ except if $x \in \mathrm{Fix}f$ so that $\mathrm{Fr}W_+$ is contained in $W_+ \cup \mathrm{Fix}f$.

As a consequence of Lemma 3, $\mathrm{int}V_0 \cap W_+ = \emptyset$ so let C be the component of $\mathbb{R} \times [-\delta, 1] \setminus \overline{W}_+$ which contains $\mathrm{int}V_0$.

Now set

$$W_{-1} = \mathrm{int}(\bigcup\{V_i \big| \ \mathrm{int}\, V_i \cap f^{-1}(\mathrm{int}\, V_0) \neq \emptyset\})$$

and

$$W_{-n} = \mathrm{int}(\bigcup\{V_i \big| \ \mathrm{int}\, V_i \cap f^{-1}(W_{-n+1}) \neq \emptyset\}), \qquad n > 1.$$

Then, as above, $W_- = \bigcup_{n\geq 1} W_{-n}$ is a connected unbounded open set.

Let Γ_+ be $\mathrm{Fr}W_+ \cap \mathrm{Fr}C = \mathrm{Adh}\widehat{W}_+ \cap \mathrm{Adh}C$, where $\widehat{W}_+$ is the union of W_+ and of all bounded components of $\mathbb{R}^2 \setminus \overline{W}_+$. By lemma 3, $W_+ \cap W_- = \emptyset$, so that $W_- \cup \mathrm{int}V_0 \subset C$ and C is unbounded. By lemma 2, $\mathrm{Fr}W_+$ is a noncompact 1-submanifold without boundary of $\mathbb{R}^2$, so that, Γ_+ being connected since W_+ and C are so, Γ_+ is a half line beginning at $(0, -\delta)$ properly embedded in $\mathbb{R} \times [-\delta, 1]$. In fact Γ_+ does not meet $\mathbb{R} \times \{1\}$, since F is a twist homeomorphism, and Γ_+ does not cross its image (as $\mathrm{Fr}W_+$). Note also that Γ_+ is composed of sides of the brick decomposition.

Now we imitate the argument in [G, §5]: Γ_+ separates $\mathbb{R} \times [-\delta, 1]$ into two open connected sets R_1 and R_2 so that (say) $f(\Gamma_+) \subset R_2 \cup \mathrm{Fix}f$. Let R be the connected component of $\bigcap_{k\in\mathbb{Z}} \tau^k(R_1)$ which contains $\mathbb{R} \times \{1\}$. Lemma 2 applies

to $\mathrm{int}(\bigcup_{k\in\mathbb{Z}}\tau^k(R_2))$ filled in by the bounded components of its complement, since $\bigcup_{k\in\mathbb{Z}}\tau^k(R_2)$ is connected. Therefore $L = \mathrm{Fr}R \cap \mathrm{Fr}(\bigcup_{k\in\mathbb{Z}}\tau^k(R_2))$ is a periodic properly embedded line in $\mathbb{R}\times\,]-\delta,1[$.

Next we show that L is contained in $(\mathbb{R}\times[-\delta,1]\setminus\overline{R})\cup \mathrm{Fix}f$: indeed, outside of any given neigborhood N of $\mathrm{Fix}f$, $\bigcap_{k\in\mathbb{Z}}\tau^k(R_1)=\bigcap_{|k|\le m}\tau^k(R_1)$ for some m, due to the finiteness modτ of the brick decomposition outside N. So that if $x\in L$, $x\notin \mathrm{Fix}f$, then $x\in\tau^k(\Gamma_+)$ for some k and $f(x)\in\tau^k(f(\Gamma_+))\subset\tau^k(R_2)\subset\mathbb{R}\times[-\delta,1]\setminus\overline{R}$.

This line L projects down in $\mathbb{R}/n\mathbb{Z}\times[-\delta,0[$ to an essential simple closed curve c such that $\tilde{F}(c)\subset \mathrm{int}c\cup \mathrm{Fix}\tilde{F}$.

Clearly such a curve cannot meet $\mathbb{R}/n\mathbb{Z}\times[-\delta,0]$ (look at a point of c realizing $d(c,\mathbb{R}/n\mathbb{Z}\times\{-\delta\})$ and use the fact that every $\mathbb{R}/n\mathbb{Z}\times\{t\}$ is preserved by $\tilde{F}$ for $-\delta\le t\le 0$ to contradict that $\tilde{F}(c)$ is essential).

To conclude the proof let us call σ the homeomorphism of $\tilde{A}=\mathbb{R}/n\mathbb{Z}\times[0,1]$ induced by the translation $(x,y)\mapsto(x+1,y)$ of the band $B=\mathbb{R}\times[0,1]$. Imitating the above argument,we consider the connected component $\tilde{R}$ of $\tilde{A}\setminus\bigcup_{0\le k\le n-1}\sigma^k(c)$ containing $\mathbb{R}/n\mathbb{Z}\times\{1\}$ and show that $\mathrm{Fr}\tilde{R}$ is a σ-equivariant closed curve which projects down onto the desired simple closed curve into $\mathrm{int}A=S^1\times\,]0,1[$.

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