PEAK SET WITHOUT PEAK POINTS

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Abstract. We give an example of a natural Banach function algebra on the
unit disc such that a smaller disc is a peak set for the algebra, but it does not
contain any peak point.

A Banach function algebra on a compact Hausdorff space $X$ is a Banach algebra
$A$ consisting of continuous functions on $X$, such that $A$ separates points of $X$ and
contains the constant functions. If the norm of the algebra $A$ coincides with the
sup norm on $X$, it is called a uniform algebra. If any linear and multiplicative
functional on $A$ is of the form $f \mapsto f(x)$ for some $x \in X$, the algebra is called
natural. A subset $K$ of $X$ is a peak set for $A$ if there is an $f \in A$ such that $f \equiv 1$
on $K$ and $|f(x)| < 1$ for $x \notin K$; if $K = \{x_0\}$, then $x_0$ is a peak point. If no proper
subset of $K$ is a peak set we call it a minimal peak set for $A$. It is well known [2]
that if $A$ is a uniform algebra on a metrizable set $X$ then any peak set contains a
peak point. In [1] H. G. Dales constructed a natural Banach function algebra on
a compact subset of $C^2$ having a peak set without any peak point. T. G. Honary
[3] provided an example in $R^3$, however his algebra is not natural. In this note we
give a very simple example of a natural Banach function algebra on the unit disc
with peak sets not containing any peak point.

Let $K$ be an open nonempty subset of the complex plane $C$. By $C^1(K)$ we denote
the algebra of all bounded complex valued functions on $K$ with continuous and
bounded first order partial derivatives on $K$. $C^1(K)$ is a Banach function algebra
on $K$ if equipped with the norm

$$
\|f\| = \|f\|_\infty + \|f_x\|_\infty + \|f_y\|_\infty,
$$

where $\|\cdot\|_\infty$ is the sup norm. By $A(K)$ we denote the uniform algebra of all continuous functions on $K$ which are analytic on $K$. We put $D_r = \{z \in C : |z| < r\}$ and $C_- = \{z : \text{Re } z < 0\}$.

Theorem 1. Let $A$ be a subalgebra of $C^1(D_1)$ consisting of all functions which are
analytic on $D_1$. Then $D_{1/2}$ is a minimal peak set for $A$; in particular $D_{1/2}$ does not
contain any peak point.

Lemma 2. There is no $h \in A(D_1)$ such that $h(1) = 0$ and $|1 + (z - 1)h(z)| < 1$
for $z \neq 1$.

Proof of the lemma. Assume that such a function does exist. If we compose $1 + (z - 1)h(z)$ with a suitable fractional linear transformation we get an $f \in A(C_-)$

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such that \( f(z) = 1 + zg(z) \), \( g(0) = 0 \), and \( |f(z)| < 1 \) for \( z \in \mathbb{C} \setminus \{0\} \). Let \( \alpha \) be a positively oriented curve consisting of the segment from \(-i\) to \(+i\) and the half of the unit circle contained in \( \mathbb{C} \). We shall show that \( \gamma = g \circ \alpha \) has negative orientation, which will contradict the fact that \( g \) is analytic.

For any \( z \in \mathbb{C} \setminus \{0\} \) we have \( \Re g(z) < 0 \), so

- as \( z \) moves up from \( 0 \) to \(+i\) along the curve \( \alpha \), \( \Im g(z) > 0 \),
- as \( z \) moves along the half circle of \( \alpha \), \( g(z) \) is in \( \mathbb{C} \setminus \{z : \Re z \leq 0, \ \Im z = 0\} \),
- as \( z \) moves from \(-i\) to \( 0 \) on the vertical line, \( \Im g(z) < 0 \).

Consequently, if \( z_0 \) is a positive real number such that the segment \((0, z_0)\) does not intersect \( \gamma \), then the orientation of \( \gamma \) around \( z_0 \), as well as around any point from \((0, z_0)\), is negative.

\[ \square \]

**Proof of the theorem.** It is well known that there is a \( C^1 \)-function, and even a \( C^\infty \)-function on \( \mathbb{D}_1 \), peaking exactly on \( \mathbb{D}^{\frac{1}{2}} \); thus \( \mathbb{D}^{\frac{1}{2}} \) is a peak set.

Assume \( \frac{1}{2} \) is a peak point for \( A \) and let \( f \in A \) be a function peaking at \( \frac{1}{2} \). Put \( u = \Re f, v = \Im f \). Since \( u^2 + v^2 \) has the maximum at \( \frac{1}{2} \), and \( u \left( \frac{1}{2} \right) = 1, v \left( \frac{1}{2} \right) = 0 \), it follows that \( u_x \left( \frac{1}{2} \right) = u_y \left( \frac{1}{2} \right) = 0 \). Since the partial derivatives are continuous, and \( f \) satisfies the Cauchy-Riemann equations on \( \mathbb{D}^{\frac{1}{2}} \), we get \( v_x \left( \frac{1}{2} \right) = v_y \left( \frac{1}{2} \right) = 0 \).

Hence

\[
g(z) = \frac{f(z) - 1}{z - \frac{1}{2}} \to 0, \quad \text{as} \quad z \to \frac{1}{2},
\]

so \( g \in A \left( \mathbb{D}^{\frac{1}{2}} \right) \), \( g \left( \frac{1}{2} \right) = 0 \), and \( f(z) = 1 + (z - \frac{1}{2}) g(z) \) has a strict maximum at \( \frac{1}{2} \).

This contradicts the lemma. Hence \( \mathbb{D}^{\frac{1}{2}} \) does not contain any peak point.

What remains to show is that \( \mathbb{D}^{\frac{1}{2}} \) does not contain any proper peak set. Assume \( K \subset \subset \mathbb{D}^{\frac{1}{2}} \) is a peak set and let \( F \in A \) be a corresponding function peaking on \( K \). Notice that \( K \) cannot contain the entire circle \( \partial \mathbb{D}^{\frac{1}{2}} \). Let \( z_0 \in K \cap \partial \mathbb{D}^{\frac{1}{2}} \) and \( z_1 \in \partial \mathbb{D}^{\frac{1}{2}} \setminus K \). For any \( w \in \partial \mathbb{D}^{\frac{1}{2}} \setminus \{z_0\} \) let \( \varphi_w \) be a \( C^1 \)-automorphism of \( \mathbb{D} \) mapping \( \mathbb{D}^{\frac{1}{2}} \) onto itself, analytic on \( \mathbb{D}^{\frac{3}{4}} \), and such that \( \varphi_w(z_0) = z_0 \) and \( \varphi_w(w) = z_1 \). Such an automorphism can be obtained by smoothly extending a suitable fractional linear automorphism of \( \mathbb{D}^{\frac{1}{2}} \). Sets of the form \( U_w = \left\{ z \in \mathbb{D}^{\frac{1}{2}} \setminus \{z_0\} : |F \circ \varphi_w(z)| < 1 \right\} \) form an open cover of \( \partial \mathbb{D}^{\frac{1}{2}} \setminus \{z_0\} \). Let \( \{w_1, w_2, \ldots\} \) be such that \( \bigcup_{j=1}^{\infty} U_{w_j} = \partial \mathbb{D}^{\frac{1}{2}} \setminus \{z_0\} \).

Put \( G = \sum_{j=1}^{\infty} \frac{F \circ \varphi_{w_j}}{\|F \circ \varphi_{w_j}\|} \). This function \( G \) belongs to \( A \) and peaks exactly at \( z_0 \), which contradicts the previous part of the proof.

\[ \square \]

Notice that essentially the same proof can be repeated to provide a slightly more general example:

Let \( \Omega_j, j = 1, 2, \ldots \), be a sequence of open subsets of \( \mathbb{D}_1 \) such that for any \( j, \Omega_j \) has an open neighborhood contained in \( \mathbb{D}_1 \) not intersecting \( \bigcup_{i \neq j} \Omega_i \), and such that \( \partial \Omega_j \) is a union of finitely many smooth Jordan curves. Let \( A \) be a subalgebra of \( C^1(\mathbb{D}_1) \) consisting of all functions which are analytic on \( \bigcup \Omega_j \). Then for any \( j, \Omega_j \) is a peak set not containing any peak point.

However, it is not clear whether the result can be further extended to Swiss cheese type domains to provide an example of a natural Banach function algebra where the union of all nontrivial minimal peak sets is dense in the unit disc. The
difficulty, in general, is not in proving that \( \Omega_j \) does not contain any peak point, but in proving that \( \overline{\Omega_j} \) is a peak set.

References


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