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FIXED POINTS OF THE MAPPING CLASS GROUP IN THE $SU(n)$ MODULI SPACES

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Abstract. Let Σ be a compact oriented surface with or without boundary components. In this note we prove that if $\chi(\Sigma) < 0$ then there exist infinitely many integers n such that there is a point in the moduli space of irreducible flat $SU(n)$ connections on Σ which is fixed by any orientation preserving diffeomorphism of Σ. Secondly we prove that for each orientation preserving diffeomorphism f of Σ and each $n \geq 2$ there is some m such that f has a fixed point in the moduli space of irreducible flat $SU(n^m)$ connections on Σ. Thirdly we prove that for all $n \geq 2$ there exists an integer m such that the m'th power of any diffeomorphism fixes a certain point in the moduli space of irreducible flat $SU(n)$ connections on Σ.

1. Introduction

Let Σ be an oriented compact surface with or without boundary components. Consider the fundamental group of the surface, π_1. Let $M(n)$ be the moduli space of irreducible flat $SU(n)$-connections on Σ. Hence we have that

$$M(n) = \mathrm{Hom}^{\mathrm{irr}}(\pi_1, SU(n))/SU(n).$$

Let Γ be the mapping class group of Σ. Let f be an element of Γ. Now f acts on $M(n)$. We are interested in the following questions:

1. Which f's have fixed points in $M(n)$?
2. Does Γ have a fixed point in $M(n)$?

We shall for certain n answer these questions affirmatively. One should remark that the fixed point set of any finite order or reducible finite order diffeomorphism can be described explicitly (see [1]). The first question is therefore only interesting for pseudo-Anosov diffeomorphisms. Answering yes to 2 *for all* n of course implies 1. It is however not quite clear if the techniques presented in this paper can establish that, though we can answer 1 positively for some nontrivial subgroups of Γ for all n.

We shall here analyse this problem using elementary finite group theory. We will show that for all primes p greater than 5 there is an integer m such that Γ has a fixed point in $M(p^m)$. We will also see that our technique can show that for all

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$n \geq 2$ there is some integer m (which may depend on n) and a point in $M(n)$ which is preserved by the m'th power of any f in Γ.

In some particular cases, a positive answer to question 1 was obtained by Frohman in [2] (see also [3]). By using some involved cohomology calculations on the classifying space of certain gauge groups, Frohman gives an existence proof for fixed points for certain diffeomorphism of surfaces with 1 boundary component.

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2. INVARIANT NORMAL SUBGROUPS OF THE FUNDAMENTAL GROUP OF A SURFACE

Let Σ be a compact oriented surface with or without boundary components. Consider the fundamental group of the surface, $\pi_1 = \pi_1(\Sigma)$. Suppose f is an orientation preserving diffeomorphism of Σ. We are now interested in f-invariant normal finite index subgroups of π_1. Suppose π is a normal subgroup of finite index of π_1. Then $f(\pi)$ is again a normal subgroup of the same finite index as π. Let $\tilde{\pi}$ be the intersection

$$\tilde{\pi} = \bigcap_{n} f^n(\pi).$$

This intersection is actually a finite intersection, since there are only finitely many normal subgroups of a given index. In fact, the number of subgroups of index i of π_1 is bounded by the number of i-fold covers of Σ. Let m be the smallest integer such that

$$\tilde{\pi} = \bigcap_{n=0}^{m-1} f^n(\pi).$$

Hence we have a finite quotient group $G_f(\pi) = \pi_1/\tilde{\pi}$, on which there is an induced action of f. If we let

$$\tilde{\pi}_k = \bigcap_{n=0}^{k} f^n(\pi),$$

then we have the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d, dash] & & \\
 & & f^{k+1}(\pi)/\tilde{\pi}_{k+1} \arrow[d, dash] & & \\
0 \arrow[r, dash] & \tilde{\pi}_k/\tilde{\pi}_{k+1} \arrow[r, dash] & \pi_1/\tilde{\pi}_{k+1} \arrow[r, dash, "q_1"] \arrow[d, dash, "q_2"] & \pi_1/\tilde{\pi}_k \arrow[r, dash] & 0. \\
 & & \pi_1/f^{k+1}(\pi) \arrow[d, dash] & & \\
 & & 0 & &
\end{tikzcd}$$

We have that (q_1, q_2) injects $\pi_1/\tilde{\pi}_{k+1}$ as a subgroup of $\pi_1/\tilde{\pi}_k \times \pi_1/f^{k+1}(\pi)$. Moreover, we also see that $\tilde{\pi}_k/\tilde{\pi}_{k+1}$ injects as a normal subgroup of $\pi_1/f^{k+1}(\pi)$, and f induces an isomorphism of $\pi_1/f^{k+1}(\pi)$ with π_1/π.

Now in order to be able to control the representation theory of $\pi_1/\tilde{\pi}$, we are going to assume that π_1/π is a simple group. We will see shortly that there are lots of examples where this is the case. The reason for this assumption is clear from the following lemma.

Lemma 2.1. *In the case where π_1/π is a simple group we have that*

$$\pi_1/\tilde{\pi} \cong \pi_1/\pi \times \pi_1/\pi \times \ldots \times \pi_1/\pi \ \textit{(m factors)}.$$

Proof. We shall prove the lemma by induction. Assume that

$$\pi_1/\tilde{\pi}_k \cong \pi_1/\pi \times \pi_1/\pi \times \ldots \times \pi_1/\pi \ (k+1 \text{ factors}).$$

This assumption is clearly true when $k = 0$. Since $\tilde{\pi}_k/\tilde{\pi}_{k+1}$ is a normal subgroup of $\pi_1/f^{k+1}(\pi)$, which is isomorphic to π_1/π, we have that $\tilde{\pi}_k/\tilde{\pi}_{k+1}$ must be either trivial or isomorphic to π_1/π. If $\tilde{\pi}_k/\tilde{\pi}_{k+1}$ is trivial, then $\tilde{\pi}_k = \tilde{\pi}$. So $k = m - 1$, in which case we have proved the lemma. Assume then that $\tilde{\pi}_k/\tilde{\pi}_{k+1} \cong \pi_1/\pi$. From the vertical sequence in the above diagram, we then see that $\pi_1/\tilde{\pi}_{k+1}$ has order $|\pi_1/\pi|^{k+1} \times |\pi_1/\pi| = |\pi_1/\pi|^{k+2}$. But then (q_1, q_2) must be an isomorphism from $\pi_1/\tilde{\pi}_{k+1}$ to $\pi_1/\tilde{\pi}_k \times \pi_1/f^{k+1}(\pi)$. By induction we then get that

$$\pi_1/\tilde{\pi}_{k+1} \cong \pi_1/\pi \times \pi_1/\pi \times \ldots \times \pi_1/\pi \ (k+2 \text{ factors}). \qquad \square$$

Remark. We note that the m in the above lemma is clearly bounded by the number of principal π_1/π bundles over Σ, which in turn is bounded by the number of $|\pi_1/\pi|$-sheeted covers of Σ.

Notice that in the above discussion, we could have defined

$$\overline{\pi} = \bigcap_{f \in \Gamma} f(\pi).$$

Exactly the same arguments show that there is some m such that

$$\pi_1/\overline{\pi} \cong \pi_1/\pi \times \pi_1/\pi \times \ldots \times \pi_1/\pi \ (m \text{ factors}).$$

This way, we get a finite group $G_\Gamma(\pi) = \pi_1/\overline{\pi}$, on which there is an induced action of the mapping class group Γ.

Using the notation from the introduction, we see that in order to construct a fixed point for Γ's action on $M(n)$, we just need to find an irreducible n-dimensional representation of the finite group $G_\Gamma(\pi)$, which is invariant under Γ. One way to construct such a situation without knowing anything about how Γ acts on $G_\Gamma(\pi)$, is to find a dimension in which $G_\Gamma(\pi)$ has only one irreducible representation. Suppose we can find a normal subgroup π of π_1 such that π_1/π is simple and has a unique irreducible representation in dimension p, where p is a prime, and such that π_1/π does not have any other irreducible representations of dimension p^n for any n. Then $G_\Gamma(\pi)$ will have a unique irreducible representation of dimension p^m. Hence we see that Γ will fix that corresponding point in $M(p^m)$.

It is now time to deliver on the promised normal subgroups.

Consider the alternating groups A_d, $d \geq 5$. These are all simple and can be given a presentation using two generators (see e.g. [5]). Since the free group on two generators is the quotient of the fundamental group of any oriented surface of negative Euler-characteristic, we see there is a normal subgroup π^d of π_1 such that

$$A_d = \pi_1/\pi^d.$$

Consider now the case where $d = p+1$, where $p > 5$ is a prime. We then have the standard irreducible representation V of A_d of dimension p. Moreover, it is easy to prove that this is the only representation of dimension p (see e.g. p.67 in [4]).[1] By using the hook length formula one can very easily prove that the symmetric group S_d does not have any irreducible representations of dimension p^n or $2p^n$ for $n > 1$. If it did, we would have that

$$p^n = \frac{(p+1)!}{h_1} \text{ or } 2p^n = \frac{(p+1)!}{h_2},$$

where here $h_1, h_2 \in \mathbb{N}$ are the hook lengths of the representations. But from this and unique factorization we get that $n = 1$. Hence A_d cannot have any irreducible representations of dimension p^n for $n > 1$. (See Proposition 5.1 p.64 in [4] for the relation between the representations of A_d and S_d.)

Hence (π^d, A_d, V) satisfies all the requirements to guarantee that there is an $m = m(d)$ which is bounded by the number of $\frac{d!}{2}$-sheeted covers of Σ, such that Γ has a fixed point in $M(p^m)$.

Hence we have proved

Theorem 2.1. *For any prime $p > 5$ there is an $m = m(p)$ such that Γ has a fixed point in $M(p^m)$.*

Now, in the case where we are just interested in one element $f \in \Gamma$, we can actually say a bit more. From the proof of lemma 2.1 we get in fact an f equivariant identification

$$G_f(\pi) \cong \pi_1/\pi \times \pi_1/f(\pi) \times \ldots \times \pi_1/f^{m-1}(\pi),$$

which means that when we use the identification

$$G_f(\pi) \cong \pi_1/\pi \times \pi_1/\pi \times \ldots \times \pi_1/\pi \text{ } (m \text{ factors}),$$

f just becomes

$$f(g_1, \ldots, g_m) = (f^m(g_m), g_1, g_2, \ldots, g_{m-1}).$$

Hence, if we have an irreducible n-dimensional representation ρ of π_1/π whose isomorphism class is preserved by f^m, then the isomorphism class of $\rho_{G_f} = \rho \otimes \ldots \otimes \rho$ is preserved by f. If ρ is the only irreducible representation of π_1/π of dimension n, the isomorphism class of ρ must be preserved by f^m.

To construct some concrete examples, consider the normal subgroup π^d of π_1 considered above. Now d is any integer greather than or equal to 5, but not equal to 6. As we mentioned before, there is only one irreducible representation of dimension $n = d-1$ of A_d. Let ρ_n be that representation. $\rho_{G_n} = \rho_n \otimes \ldots \otimes \rho_n$ then gives a fixed point for f in $M(n^m)$.

We can alternatively fix the normal subgroup π of π_1 and then change f to a power of f. We no longer have to worry about what f maps π to, since we will simply replace f by a power of f which fixes π. The requirement that π_1/π should

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[1] This is not the case for A_6, so we have excluded this case.

be simple is no longer needed. We can for example make the following construction. Let π be a normal subgroup of π_1 such that

$$\pi_1/\pi = S_3.$$

We then know that for some m less than the number of 6-sheeted covers of Σ, we have

$$f^m(\pi) = (\pi).$$

Now S_3 has a unique unitary irreducible 2-dimensional representation. Then this reprentation gives us a point in $M(2)$ which is fixed by f^m for all $f \in \Gamma$. (We have here used the same symbol $M(2)$ to represent the $U(2)$ and the $SU(2)$ moduli space.) The smallest m that does this will of course depend on f. However, if we let l be the number of normal subgroups of π_1 whose quotient is S_3, then f^l preserves the given point in $M(2)$.

Now notice that each f in Γ permutes the l normal subgroups with S_3 quotient; hence we get a homomorphism from Γ to S_l. The kernel of this homomorphism is of course a normal subgroup of finite index of Γ. For the elements f in this normal subgroup, we can by definition take $m = 1$ above, i.e. they fix the above described point in $M(2)$.

To get such fixed points in the moduli spaces $M(n)$ where n is not 2, we just repeat the above construction, where we replace S_3 with some group which has a unique irreducible representation in dimension n. For $n \neq 5$ just take the unique irreducible representation of A_{n+1} of dimension n. For $n = 5$ take the unique irreducible 5-dimensional representation of A_4.

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