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# NEW RAPIDLY CONVERGENT SERIES REPRESENTATIONS FOR $\zeta(2n+1)$

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Abstract. We give three series representations for the values of the Riemann zeta function $\zeta(s)$ at positive odd integers. One representation extends Ewell's result for $\zeta(3)$ [Amer. Math. Monthly **97** (1990), 219–220] and is considerably simpler than the two generalisations proposed earlier. The second representation is even simpler:

$$\zeta(2n+1) = (-1)^n \frac{4(2\pi)^{2n}}{(2n+1)!} \sum_{k=0}^{\infty} R_{2n+1,k}\zeta(2k), \qquad n \geq 1,$$

where the coefficients $R_{2n+1,k}$ for a fixed $n$ are rational in $k$ and are explicitly given by the finite sum involving the Bernoulli numbers. The third representation is obtained from the second by the Kummer transformation. We demonstrate the rapid convergence of this series using several examples.

## 1. Introduction

The Riemann zeta function $\zeta(s)$ is defined for $\operatorname{Re} s > 1$ as [1, p. 19]

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}. \tag{1}$$

For $\operatorname{Re} s \leq 1$, $s \neq 1$, $\zeta(s)$ is defined as the analytic continuation of (1) and is analytic over the whole complex plane, except at $s = 1$, where it has a simple pole.

Recall that $\zeta(0) = -1/2$ and that for $n = 1, 2, 3, \ldots$ we have [1, p. 19]

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \tag{2a}$$

$$\zeta(-2n) = 0, \qquad \zeta(-2n+1) = -\frac{B_{2n}}{2n}, \tag{2b}$$

where the rational numbers $B_m$ are the Bernoulli numbers. The celebrated formula in (2a), derived by Euler in 1740, expresses $\zeta(2n)$ as a rational multiple of $\pi^{2n}$. There is no analogous closed evaluation for $\zeta(2n+1)$, and various series and integral representations have been derived. Probably the most famous is the formula stated twice without proof by Ramanujan [2, Vol. I, p. 259, No. 15 and Vol. II, p. 117, No. 21]. This result was proved by several mathematicians (notably by Berndt [3]

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who also gives a historical account of the formula) and various analogous formulae, special cases and generalizations have been established (for a detailed bibliography see [4, pp. 275–276]).

Another remarkable result, Apéry's proof of the irrationality of $\zeta(3)$ based on the rapidly converging series (see [5] and [6] and the essay in [7])

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^3\binom{2k}{k}}, \tag{3}$$

caused a sensation at the International Congress of Mathematicians in Helsinki in 1978. There exist related simple formulae for $\zeta(2)$ and $\zeta(4)$, but expressions for $\zeta(2n+1)$, $n \geq 2$, are much more complicated [6, 8, 9] (see Section 4).

Ewell [10] found a new simple series

$$\zeta(3) = -\frac{4\pi^2}{7}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}} \tag{4}$$

and, in an attempt to generalise the result, showed that there exists a multiple series representation of $\zeta(n)$ in the form

$$\zeta(n) = \frac{2^{n-2}}{2^n-1}\pi^2\sum_{m=0}^{\infty}(-1)^m A_{2m}(n-2)\pi^{2m}/(2m+2)! \tag{5}$$

for every integer $n > 2$ [11]. The coefficients $A_{2m}(n)$ are multiple finite sums which involve multinomial coefficients and the Bernoulli numbers. For $n = 3$ the formula in (5) reduces to (4). However, with increasing $n$ the coefficients $A_{2m}(n)$ become so complicated that it does not seem possible to arrive even at $\zeta(4) = \pi^4/90$. Ewell's method was modified by Yue and Williams [12], who deduced several series analogous to that in (4) and obtained a new formula for $\zeta(2n+1)$, $n \geq 2$. Although still complicated, their representation is simpler than that given by Ewell. Finally, note that Ewell [13] recently deduced a new series representation of the values $\zeta(2n+1)$ in a determinantal form.

We give three series representations for $\zeta(2n+1)$, $n \geq 1$. Theorem A uses the method of Yue and Williams and extends and generalises Ewell's result in (4). We demonstrate that it leads to a much simpler series than they actually obtained. Theorem B adopts a different approach, based on an integral representation for $\zeta(2n+1)$, and results in an even further simplification. As a corollary to Theorem B we obtain a third representation by making use of the Kummer transformation. All three representations are very simple, do not become complicated with increasing $n$, and converge very rapidly.

## 2. Statement of results

In what follows the $n$th Bernoulli number $B_n$ and the Bernoulli polynomial $B_n(x)$ of degree $n$ are defined by [1, p. 25]

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!} \quad \text{and} \quad \frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{6}$$

respectively, when $|t| < 2\pi$. Note that $B_n = B_n(0)$. At times, we use the Pochhammer symbol $(s)_n$ given by

$$(s)_0 = 1, \qquad (s)_n = s(s+1)(s+2)\cdots(s+n-1), \quad n = 1,2,3,\ldots. \tag{7}$$

The results are as follows.

**Theorem A.** *Let $\zeta(s)$ be the Riemann zeta function and $n$ a positive integer. We then have*

$$\zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n}}{n(2^{2n+1}-1)} \left[ \sum_{k=1}^{n-1} (-1)^{k-1} \frac{k\zeta(2k+1)}{\pi^{2k}(2n-2k)!} + \sum_{k=0}^{\infty} \frac{\zeta(2k)(2k)!}{2^{2k}(2k+2n)!} \right]$$

*where the finite sum on the right-hand side is* 0 *when* $n = 1$.

**Theorem B.** *Let $n$ be a positive integer. We have*

$$\zeta(2n+1) = (-1)^n \frac{4(2\pi)^{2n}}{(2n+1)!} \sum_{k=0}^{\infty} R_{2n+1,k}\,\zeta(2k)$$

*where the coefficients $R_{2n+1,k}$ are rational and given by*

$$\text{(i)} \qquad R_{2n+1,k} = \sum_{m=0}^{2n} \binom{2n}{m} \frac{(2n+1)B_{2n-m}}{2^{2k+m+1}(2k+m+1)(m+1)}, \qquad k = 0,1,2,\ldots,$$

*and by the family of generating functions*

$$\text{(ii)} \qquad \sum_{n=0}^{\infty} R_{2n+1,k} \frac{t^{2n+1}}{(2n+1)!} = \frac{(2k-1)!}{t^{2k-1}(e^t-1)} \left( 1 - e^{t/2} \sum_{m=0}^{2k-1} (-1)^m \frac{t^m}{m!\,2^m} \right), \qquad k = 1,2,3,\ldots,$$

*when* $|t| < 2\pi$.

**Corollary.** *Let $n$ be a positive integer. We have*

$$\zeta(2n+1) = (-1)^n \frac{4(2\pi)^{2n}}{(2n+1)!} \left[ \sum_{k=0}^{\infty} R_{2n+1,k} + \sum_{k=0}^{\infty} R_{2n+1,k}(\zeta(2k)-1) \right]$$

*where the coefficients $R_{2n+1,k}$ have the same meaning as in Theorem B.*

**Note 1.** Observe that, for a fixed $n$, on using the expressions in Theorem B (i) or the generating function in (14) below, $R_{2n+1,k}$ is readily available as

$$R_{2n+1,k} = \frac{1}{2^{2k}} \frac{P_n(k)}{(2k+2n)\prod_{j=0}^{n}(2k+2j+1)}, \qquad n = 1,2,3,\ldots,$$

where $P_n(k)$ is a polynomial in $k$ of degree $n$ with rational coefficients. Once deduced for a particular $n$ (see examples in Section 4), this form is then valid for $k = 0,1,2,\ldots$. Moreover, closed-form summation of the series $\sum_{k=0}^{\infty} R_{2n+1,k}$ in the representation given by the Corollary is then always possible. For example,

$$\sum_{k=0}^{\infty} R_{3,k} = -\frac{1}{2} + \frac{3}{2}\log\left(\frac{3}{2}\right), \qquad \sum_{k=0}^{\infty} R_{5,k} = -\frac{17}{16} + \frac{5}{2}\log\left(\frac{3}{2}\right).$$

**Note 2.** In view of the Euler relation in (2a), two particularly elegant formulae

$$\zeta(2n+1) = \zeta(2n)\sum_{k=0}^{\infty} r_{2n+1,k}\zeta(2k) \quad \text{and} \quad \zeta(2n+1) = \zeta(2n)\sum_{k=0}^{\infty} \rho_{2n+1,k}\pi^{2k}$$

follow from Theorem B. The coefficients $r_{2n+1,k}$ and $\rho_{2n+1,k}$ are rational and given by

$$r_{2n+1,k} = \frac{-8}{(2n+1)B_{2n}} R_{2n+1,k} \quad \text{and} \quad \rho_{2n+1,k} = (-1)^k \frac{4\,2^{2k}B_{2k}}{(2n+1)B_{2n}(2k)!} R_{2n+1,k}.$$

## 3. Proof of the results

Theorem A will be established by making use of the functional equation for the Riemann zeta function $\zeta(s)$ [1, p. 19],

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2}, \tag{8a}$$

and the summation formula [12, p. 1587, Eq. (3.5)]

$$(2^s - 2)\zeta(s) = \sum_{k=1}^{\infty} \frac{(s+1)_{2k}\zeta(s+2k)}{2^{2k}(2k)!} \tag{8b}$$

where by (7) $(s+1)(s+2)\cdots(s+2k-1)$ is written as $(s+1)_{2k}$.

First, we combine the formulae in (8a) and in (8b), with the result

$$\begin{aligned} (2^{2s} - 2^{s+1})\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2} &= \sum_{k=1}^{\infty} \frac{(s+1)_{2k}\zeta(s+2k)}{2^{2k}(2k)!} \\ &= \sum_{k=1}^{n-1} \frac{(s+1)_{2k}\zeta(s+2k)}{2^{2k}(2k)!} + \sum_{k=n}^{\infty} \frac{(s+1)_{2k}\zeta(s+2k)}{2^{2k}(2k)!}. \end{aligned} \tag{9}$$

Assuming for the moment that $n \geq 2$ and dividing the left-hand and right-hand sides of (9) by $(s+1)_{2n} = (s+1)_{2n-1}(s+2n)$, we have

$$\begin{aligned} S &= (2^{2s} - 2^{s+1})\pi^{s-1}\Gamma(1-s)\zeta(1-s)\frac{1}{(s+1)_{2n-1}}\frac{\sin(\pi s/2)}{(s+2n)}, \\ S_1 &= \sum_{k=1}^{n-1} \frac{1}{2^{2k}(2k)!}\frac{(s+1)_{2k}}{(s+1)_{2n-1}}\frac{\zeta(s+2k)}{(s+2n)}, \\ S_2 &= \sum_{k=n}^{\infty} \frac{(s+2n+1)\cdots(s+2k)}{2^{2k}(2k)!}\zeta(s+2k) \end{aligned}$$

where $S = S_1 + S_2$. Next, it is evident that

$$\lim_{s\to-2n}\frac{\sin(\pi s/2)}{s+2n} = \frac{(-1)^n\pi}{2} \quad \text{and} \quad \lim_{s\to-2n}\frac{\zeta(s+2k)}{s+2n} = \zeta'(-2(n-k)) \tag{10a}$$

where $\zeta'(\ )$ stands for the first derivative of $\zeta(\ )$. However, by (8a) we have

$$\begin{aligned} \zeta'(-2(n-k)) &= \lim_{\delta\to 0}\frac{\zeta(-2n+2k+\delta)}{\delta} \\ &= \frac{(-1)^{n-k}}{2(2\pi)^{2(n-k)}}(2(n-k))!\zeta(2(n-k)+1). \end{aligned} \tag{10b}$$

Finally, let $s \to -2n$. Then, in view of (10a) and (10b), and from the properties of the Pochhammer symbol (see, for instance, [14, Chapter 18]) and of the sums, it is seen that

$$\begin{aligned}
S_1 &= \sum_{k=1}^{n-1} \frac{1}{2^{2k}(2k)!} \frac{(1-2n)_{2k}}{(1-2n)_{2n-1}} \zeta'(-2(n-k)) \\
&= -\sum_{k=1}^{n-1} \frac{\zeta'(-2(n-k))}{2^{2k}(2k)!(2(n-k)-1)!} \\
&= \frac{1}{2^{2n}} \sum_{k=1}^{n-1} (-1)^{k-1} \frac{k\zeta(2k+1)}{\pi^{2k}(2n-2k)!}, \\
S_2 &= \sum_{k=n}^{\infty} \frac{(-2n+2k)!}{2^{2k}(2k)!} \zeta(-2n+2k) = \frac{1}{2^{2n}} \sum_{k=0}^{\infty} \frac{\zeta(2k)(2k)!}{2^{2k}(2k+2n)!}, \\
S &= (-1)^{n-1} \frac{(2^{2n+1}-1)(2n)!\zeta(2n+1)}{2\pi^{2n}2^{4n}} \frac{1}{(1-2n)_{2n-1}} \\
&= (-1)^n \frac{n(2^{2n+1}-1)\zeta(2n+1)}{\pi^{2n}2^{4n}}.
\end{aligned}$$

It remains to verify that $S = S_1 + S_2$ gives the desired result. Note that the result obtained assuming $n \geq 2$ is also valid for $n = 1$. This completes the proof.

*Proof of Theorem B.* The proof makes use of two familiar results: the representation of the Bernoulli polynomials $B_{2n+1}(x)$ by the trigonometric series [1, p. 27]

(11a)

$$B_{2n+1}(x) = (-1)^{n+1} \frac{2(2n+1)!}{(2\pi)^{2n+1}} \sum_{m=1}^{\infty} \frac{\sin(2m\pi x)}{m^{2n+1}} \qquad (0 \leq x \leq 1; n = 1, 2, 3, \dots)$$

and the following expansion for the cotangent [1, p. 36]:

$$\pi x \cot(\pi x) = -2 \sum_{m=0}^{\infty} \zeta(2m) x^{2m}, \qquad |x| < 1. \tag{11b}$$

We first show that

$$\zeta(2n+1) = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{(2n+1)!} \int_0^{1/2} B_{2n+1}(t) \cot(\pi t)\, dt, \qquad n = 1, 2, 3, \dots. \tag{12}$$

Observe that the existence of the integral in (12) is assured since

$$\lim_{t \to 0} B_{2n+1}(t) \cot(\pi t) = \frac{1}{\pi} \binom{2n+1}{2n} B_{2n}.$$

Indeed, from the Fourier expansion of $B_{2n+1}(t)$ given in (11a), and by inverting the order of summation and integration by absolute convergence, we have

$$(-1)^{n+1} \frac{(2\pi)^{2n+1}}{(2n+1)!} \int_0^{1/2} B_{2n+1}(t) \cot(\pi t)\, dt = 2 \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \int_0^{1/2} \sin(2k\pi t) \cot(\pi t)\, dt.$$

For $k = 1, 2, 3, \dots$ the integral on the right-hand side exists (the integrand has a removable singularity at $t = 0$) and is constant,

$$\int_0^{1/2} \sin(2k\pi t)\cot(\pi t)\,dt = \frac{1}{2\pi}\left(\int_0^{\pi/2} \frac{\sin(2k+1)t}{\sin t}\,dt + \int_0^{\pi/2} \frac{\sin(2k-1)t}{\sin t}\,dt\right) = \frac{1}{2},$$

since it readily follows from

$$\frac{\sin(2m+1)t}{\sin t} = 1 + 2\sum_{k=1}^{m} \cos(2kt)$$

that

$$\int_0^{\pi/2} \frac{\sin(2m+1)t}{\sin t} dt = \frac{\pi}{2}$$

for any integer $m \geq 0$. In this way we arrive at the formula proposed in (12).

Next, the integral representation of $\zeta(2n+1)$ in (12) in conjunction with the expansion of the cotangent in (11b) gives the series representation

$$\zeta(2n+1) = (-1)^n \frac{4(2\pi)^{2n}}{(2n+1)!} \sum_{k=0}^{\infty} R_{2n+1,k}\,\zeta(2k) \tag{13a}$$

where

$$R_{2n+1,k} = \int_0^{1/2} B_{2n+1}(t)\,t^{2k-1}\,dt \qquad (n = 1, 2, 3, \dots; k = 0, 1, 2, \dots). \tag{13b}$$

The inversion of the order of summation and integration in (13a) and (13b) is justified by absolute convergence. Finally, knowing that the polynomials $B_{2n+1}(t)$ do not contain a free term, in view of (13b) it is clear that $R_{2n+1,k}$ assumes rational values.

(i) The result follows upon using in (13b) the explicit definition

$$B_n(x) = \sum_{m=0}^{n} \binom{n}{m} B_m x^{n-m}, \qquad n = 0, 1, 2, \dots,$$

of the Bernoulli polynomials [1, p. 25] and further simplification using the properties of the binomial coefficients and sums.

(ii) By multiplying the generating relation for the Bernoulli polynomials in (6) by $x^{2\kappa-1}$ ($\operatorname{Re}\kappa > 0$) and integrating over $(0,1/2)$ we deduce that

$$\begin{aligned} G(t,\kappa) &= \frac{t}{e^t-1}\int_0^{1/2} e^{tx}x^{2\kappa-1}dx = \frac{t}{e^t-1}\sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^{1/2} x^{m+2\kappa-1}dx \\ &= \frac{t}{e^t-1}\sum_{m=0}^{\infty}\frac{1}{(2\kappa+m)m!2^{2\kappa}}\left(\frac{t}{2}\right)^m \\ &= \frac{1}{2}\frac{t}{(e^t-1)\kappa 2^{2\kappa}}\sum_{m=0}^{\infty}\frac{(2\kappa)_m}{(2\kappa+1)_m}\frac{(t/2)^m}{m!} \\ &= \frac{1}{2}\frac{t}{(e^t-1)\kappa 2^{2\kappa}}\,{}_1F_1\left[\begin{matrix}2\kappa \\ 2\kappa+1\end{matrix}\; t/2\right], \qquad \operatorname{Re}\kappa > 0, \end{aligned} \tag{14a}$$

is the exponential generating function of $R_{n,\kappa}$,

$$R_{n,\kappa} = \int_0^{1/2} B_n(x)x^{2\kappa-1}\,dt \qquad (n = 1,2,3,\ldots;\operatorname{Re}\kappa > 0), \tag{14b}$$

since

$$G(t,\kappa) = \sum_{n=0}^{\infty} R_{n,\kappa}\frac{t^n}{n!}, \qquad |t| < 2\pi. \tag{14c}$$

It follows without difficulty that the expansion in (14c) is valid when $|t| < 2\pi$. When $\kappa$ is a positive integer, the Kummer hypergeometric function ${}_1F_1$ [1, Chapter VI] involved in $G(t,\kappa)$, can be simplified by [15, p. 579, Entry 7.11.1.13]

$${}_1F_1\left[\begin{matrix}n \\ n+1\end{matrix}\; z\right] = (-1)^n\frac{n!}{z^n}\left(1 - e^z\sum_{m=0}^{n-1}(-1)^m\frac{z^m}{m!}\right), \qquad n = 1,2,3,\ldots,$$

thus resulting in the desired generating function for the numbers $R_{n,k}$.

*Proof of the Corollary.* We use the Kummer transformation of series [16, p. 247]. Let $\sum_{k=0}^{\infty} a_k = \alpha$ and $\sum_{k=0}^{\infty} b_k = \beta$ be convergent series. If the terms of these two series are asymptotically proportional, namely $\lim_{k\to\infty}\frac{a_k}{b_k} = L \neq 0$, then

$$\alpha = \beta L + \sum_{k=0}^{\infty}\left(1 - \frac{b_k}{a_k}L\right)a_k.$$

Note that, in view of the absolute convergence of the series, the validity of our corollary is trivial. However, we are interested in the consequence of the Kummer transformation, that the transformed series, here $\sum_{k=0}^{\infty} R_{2n+1,k}(\zeta(2k)-1)$, converges more rapidly than the original series $\sum_{k=0}^{\infty} R_{2n+1,k}\zeta(2k)$.

In our case, for sufficiently large $k$ and $n \geq 1$ we have (see Note 1)

$$|R_{2n+1,k}| = \frac{1}{2^{2k}}\left|\frac{P_n(k)}{(2k+2n)\prod_{j=0}^{n}(2k+2j+1)}\right| \leq \frac{1}{2^{2k}}$$

since $P_n(k)$ is a polynomial in $k$ of degree $n$, and therefore the series $\sum_{k=0}^{\infty} R_{2n+1,k}$ is absolutely convergent by the comparison test. On putting $a_k = R_{2n+1,k}\zeta(2k)$, $b_k = R_{2n+1,k}$ and $L = 1$ (since for large $k$ we have $\zeta(2k) \to 1$), we arrive at the proposed formula.

## 4. Examples and concluding remarks

Two particularly simple series for $\zeta(3)$, different from the defining series, are those of Apéry and Ewell given in (3) and in (4), respectively. Apéry's series is generalised to [6, 8, 9, 17]

$$\zeta(2n+1) = 2\sum_{k=1}^{n}\sum_{j=k}^{\infty}\frac{t(2j,2k)}{j^{2n-2k+1}(2j)!} + \frac{1}{2}\sum_{j=n}^{\infty}\frac{t(2j,2n)}{j(2j)!}, \qquad n \geq 1, \tag{15}$$

where $t(j,k)$ are the central factorial numbers, defined as the coefficients of the expansion

$$\left[2\,\mathrm{arcsinh}\,\frac{t}{2}\right]^k = k!\sum_{j=k}^{\infty} t(j,k)\frac{t^j}{j!} \qquad (|t| < 2; k \in N_0).$$

By using $t(j,k)$ the series representation in (15) assumes a more compact form than that given by Leshchiner [8].

The generalised Ewell series representation is given by Theorem A. For $n = 1$ it gives the result in (4). When $n = 2$ and 3 we obtain

$$\zeta(5) = \frac{4\pi^2}{31}\zeta(3) + \frac{8\pi^4}{31}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+1)(2k+2)\cdots(2k+4)} \tag{16a}$$

and

(16b)

$$\zeta(7) = -\frac{8\pi^4}{1143}\zeta(3) + \frac{64\pi^2}{381}\zeta(5) - \frac{64\pi^6}{381}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+1)(2k+2)\cdots(2k+6)},$$

respectively. On the other hand, from Theorem B we deduce that

$$\zeta(3) = -\frac{\pi^2}{3}\sum_{k=0}^{\infty}\frac{(2k+5)\zeta(2k)}{2^{2k}(2k+1)(2k+3)(2k+2)}, \tag{17a}$$

$$\zeta(5) = -\frac{\pi^4}{180}\sum_{k=0}^{\infty}\frac{(28k^2+168k+269)\zeta(2k)}{2^{2k}(2k+1)(2k+3)(2k+5)(2k+4)}, \tag{17b}$$

$$\zeta(7) = -\frac{\pi^6}{7560}\sum_{k=0}^{\infty}\frac{(248k^3+2604k^2+9394k+11757)\zeta(2k)}{2^{2k}(2k+1)(2k+3)(2k+5)(2k+7)(2k+6)}. \tag{17c}$$

Finally, as an example of the application of the Corollary we have

$$\zeta(3) = \frac{4\pi^2}{3}\left[1 - 3\log\left(\frac{3}{2}\right)\right] + \frac{\pi^2}{3}\sum_{k=0}^{\infty}\frac{(2k+5)(1-\zeta(2k))}{2^{2k}(2k+1)(2k+3)(2k+2)}. \tag{18}$$

Note that the connection between the series obtained in Theorem A, those given by Wilton [18, p. 92] and the series recently obtained by Tsumura [19, Theorem B] is apparently less closed than might be anticipated. The rapid convergence of the series representation for $\zeta(2n+1)$ proposed in this work is demonstrated by several examples given in Table 1. Note that, with $n_0 = 11$ and $n_0 = 101$, Apéry's series in (3) carries the error of $4.44 \times 10^{-10}$ and $1.32 \times 10^{-66}$, respectively.

## Note added in proof

A paper on a similar subject has recently appeared (A. Dabrowski, Nieuw Arch. Wisk. (4) **14** (1996), 119–207).

TABLE 1. Errors in the computation of the Riemann zeta function when the first $n_0$ terms of the various series are summed. The number of terms of the defining series given in (1) required for the same accuracy is given in brackets.

| $n_0$ | series | $\zeta(3)$ | $\zeta(5)$ | $\zeta(7)$ |
|---|---|---|---|---|
| 11 | defining series | $3.77 \times 10^{-3}$ | $1.42 \times 10^{-5}$ | $7.11 \times 10^{-8}$ |
| | Theorem B | $1.95 \times 10^{-9}$ $(1.60 \times 10^{4})$ | $2.23 \times 10^{-9}$ (103) | $2.32 \times 10^{-9}$ (21) |
| | Corollary | $3.86 \times 10^{-16}$ $(3.60 \times 10^{7})$ | $4.42 \times 10^{-16}$ (4887) | $4.60 \times 10^{-16}$ (268) |
| 101 | defining series | $4.85 \times 10^{-5}$ | $2.36 \times 10^{-9}$ | $1.52 \times 10^{-13}$ |
| | Theorem B | $1.65 \times 10^{-65}$ $(1.74 \times 10^{32})$ | $1.9 \times 10^{-65}$ $(1.07 \times 10^{16})$ | $1.98 \times 10^{-65}$ $(4.51 \times 10^{10})$ |
| | Corollary | $2.07 \times 10^{-126}$ $(4.92 \times 10^{62})$ | $2.38 \times 10^{-126}$ $(1.80 \times 10^{31})$ | $2.48 \times 10^{-126}$ $(6.38 \times 10^{20})$ |

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