

## ***K*-REGULAR WITT RINGS**

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ABSTRACT. We improve Kula's bounds on the size of possible  $k$ -regular Witt rings.

$(R, G, q)$  will denote an abstract Witt ring in the sense of [4]. Nearly all examples of interest are Witt rings of non-singular quadratic forms over a field of characteristic not two; however, using abstract Witt rings does simplify some proofs. The Witt ring is  $k$ -regular if there exists a 2-power  $k$  such that for all  $1 \neq x \in G$  we have  $|D(1, -x)| = k$ . Such Witt rings were first studied in [1] primarily because the block design counting arguments there were perfectly suited to  $k$ -regular rings. However they remain unclassified.

We will always assume that  $G$  is finite; set  $g = |G|$ . If  $k = g$ , then  $R$  is totally degenerate and so classified by [4]. If  $k = g/2$ , then  $R$  is of local type [2] which is again classified in [4]. If  $k = 2$ , then  $R$  is a group ring extension of  $\mathbb{Z}_2$  or  $\mathbb{Z}_4$ . If  $2 < k < g/2$ , then  $R$  is not of elementary type and no examples are known or even expected. We will always assume that  $2 < k < g/2$  and call such  $k$ -regular Witt rings *exceptional*.

It was shown in [1] that exceptional  $k$ -regular Witt rings satisfy  $8 \leq k$  and  $2k^2 \leq g$ . Kula [3] improved both bounds and added an upper bound, showing:

$$\begin{aligned} 16 &\leq k, \\ 8k^2 &\leq g \leq k^4/4 \quad \text{if } k \equiv 1 \pmod{3}, \\ 8k^2 &\leq g \leq k^4/8 \quad \text{if } k \equiv 2 \pmod{3}. \end{aligned}$$

Here we show that  $k^3 \leq g$  and that if  $k \equiv 1 \pmod{3}$  then  $g \equiv 1 \pmod{3}$ .

We fix some notation, which will agree with Kula's.  $G^*$  denotes  $G \setminus \{1\}$ . We set  $e = \log_2 k$ . For  $a \in G^*$  and  $i \geq 0$  set:

$$X_i(a) = \{x \in G: x \neq 1, a \text{ and } |Q(a) \cap Q(x)| = 2^i\},$$

where  $Q(x) = \{q(x, y): y \in G\}$ . Now for  $x \neq a$ ,

$$|Q(a) \cap Q(x)| = |D\langle 1, -ax \rangle| / |D\langle 1, -a \rangle \cap D\langle 1, -x \rangle| = k / |D\langle 1, -a \rangle \cap D\langle 1, -x \rangle|.$$

Thus we also have that:

$$X_i(a) = \{x \in G: x \neq 1, a \text{ and } |D\langle 1, -a \rangle \cap D\langle 1, -x \rangle| = 2^{e-i}\}.$$

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In particular, we may assume  $0 \leq i \leq e$ . We further set  $n_i(a) = |X_i(a)|$  and write  $X(a)$  for  $X_e(a)$ . For a 2-fold Pfister form  $\rho$  we let  $\rho'$  denote the pure part of  $\rho$ .

We will use various equations derived by Kula:

$$(1) \quad \sum_{i=0}^{e-1} (2^{e-i} - 1)n_i(a) = k^2 - 3k + 2,$$

$$(2) \quad g + \sum_{1 \neq \rho \in Q(a)} |D(\rho')| = 1 + \frac{g}{k} + \sum_{i=0}^e 2^i n_i(a),$$

$$(3) \quad |X(a) \cap X(b)| \geq g - 2k^2 + 6k - 7 \geq g - 2k^2,$$

where  $a \neq b$  in  $G^*$  for (3). Equation (1) is [3, 4.3b], (2) is equation (4.5.2) in [3, p. 45] and the first inequality of (3) is equation (4.3.1) in [3, p. 43]. The second inequality of (3) follows from our assumption that  $k > 2$ .

We will also use two simple equations:

$$(4) \quad \sum_{i=0}^e n_i(a) = g - 2,$$

$$(5) \quad |D(\rho')| < k^2 \quad (\text{if } \rho \neq 1).$$

Both (4) and (5) appear in [3] but direct proofs are quick. (4) follows from  $G \setminus \{1, a\}$  being the union of the  $X_i(a)$ . For (5), suppose  $\rho' = \langle a, b, ab \rangle$ . Then

$$D(\rho') = a \cdot \bigcup_{x \in D\langle 1, a \rangle} D\langle 1, bx \rangle.$$

Since 1 occurs in each  $D\langle 1, bx \rangle$  we have that  $|D(\rho')| < |D\langle 1, a \rangle| \cdot k = k^2$ .

Using equation (4) to find  $n_e(a)$  and equation (1) to find  $n_{e-1}(a)$ , equation (2) may be re-written (see [3, pp. 45–46]) as:

$$(6) \quad g + \sum_{1 \neq \rho \in Q(a)} |D(\rho')| = 1 + \frac{g}{k} + gk - \frac{k^3}{2} + \frac{3k^2}{2} - 3k + \sum_{i=0}^{e-2} 2^i (2^{e-i-1} - 1)(2^{e-i} - 1)n_i(a).$$

**Proposition 1.** *If  $k \equiv 1 \pmod{3}$ , then  $g \equiv 1 \pmod{3}$ .*

*Proof.* We may pick an  $a \in G^*$  with  $\langle\langle 1, 1 \rangle\rangle \notin Q(a) \setminus \{1\}$  (otherwise  $-G^* \subset D\langle 1, 1, 1 \rangle$  while  $|D\langle 1, 1, 1 \rangle| < k^2$  by (5) and  $|G^*| \geq 8k^2 - 1$  by [3, 4.4]). Then for each anisotropic  $\rho \in Q(a)$  we have that  $|D(\rho')| \equiv 0 \pmod{3}$  by [3, 2.9]. Also, since for each  $i$ , in equation (6) one of  $e - i - 1$  or  $e - i$  is even, we have that one of  $2^{e-i-1} - 1$  or  $2^{e-i} - 1$  is divisible by 3. Assuming  $k \equiv 1 \pmod{3}$ , equation (6) gives:

$$g \equiv g + 1 + g - 2 \pmod{3},$$

and so  $1 \equiv g \pmod{3}$ . □

**Theorem 1.**  $g \geq k^3$ .

*Proof.* Suppose there exists an exceptional  $k$ -regular Witt ring  $(R, G)$  with  $g < k^3$ . Among all such Witt rings, choose one with minimal  $h \equiv g/k^2$ . Let  $a$  and  $b$  be distinct elements of  $G^*$ . Choose  $x \in X(a) \cap X(b)$ , which is possible by equation (3) and the fact that  $g \geq 8k^2$  [3, 4.4]. We use the equation (4.3.2) from [3, p. 43]:

$$\begin{aligned}
 (7) \quad hk = g/k = |Q(x)| &\geq |(Q(x) \cap Q(a))(Q(x) \cap Q(b))| \\
 &= \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} \\
 &\geq \frac{k^2}{|Q(a) \cap Q(b)|} = k|D\langle 1, -a \rangle \cap D\langle 1, -b \rangle|.
 \end{aligned}$$

A simple consequence of (7) is that  $|D\langle 1, -a \rangle \cap D\langle 1, -b \rangle| \leq h$ . Pick a minimal  $s \geq 0$  so that there exist distinct  $a$  and  $b$  in  $G^*$  with  $|D\langle 1, -a \rangle \cap D\langle 1, -b \rangle| = h/2^s$ . Set  $2^t = |Q(a) \cap Q(b)|$ . Then we have:

$$(8) \quad g \geq 2^{s+2}k^2 \quad \text{and} \quad t - s \geq 1.$$

Namely, if the first inequality failed, then  $h = g/k^2 \leq 2^{s+1}$ . But then  $|D\langle 1, -a \rangle \cap D\langle 1, -b \rangle| \leq 2$  for all distinct  $a$  and  $b$  in  $G^*$ , while as noted in the first sentence of [3, p. 44] we can always find distinct  $a$  and  $b$  in  $G^*$  with  $|D\langle 1, -a \rangle \cap D\langle 1, -b \rangle| \geq 4$ . For the second inequality of (8) note that:

$$2^t = |Q(a) \cap Q(b)| = \frac{k}{|D\langle 1, -a \rangle \cap D\langle 1, -b \rangle|} = \frac{2^s k}{h}.$$

Thus  $2^{t-s}h = k$ . By the assumption that  $g = hk^2 < k^3$  we have  $2h \leq k$  and so  $t - s \geq 1$ .

For each  $x \in X(a) \cap X(b)$  we can rewrite (7) as:

$$(9) \quad hk \geq \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} \geq \frac{k^2}{|Q(a) \cap Q(b)|} = \frac{hk}{2^s}.$$

Then

$$|Q(x) \cap Q(a) \cap Q(b)| \geq 2^{t-s}$$

since otherwise  $|Q(x) \cap Q(a) \cap Q(b)| < 2^{t-s} = |Q(a) \cap Q(b)|/2^s$  and equation (9) becomes:

$$hk \geq \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} > \frac{2^s k^2}{|Q(a) \cap Q(b)|} = hk.$$

List the elements of  $Q(a) \cap Q(b)$  as  $1, \rho_2, \dots, \rho_{2^t}$ . We have for each  $x \in X(a) \cap X(b)$  that  $2^{t-s} - 1$  of the  $\rho_i$ 's lie in  $Q(x)$ , or equivalently, satisfy  $-x \in D(\rho'_i)$ . Set  $T_x$  equal to the number of  $i$ 's,  $2 \leq i \leq 2^t$ , such that  $-x \in D(\rho'_i)$ . Then:

$$(10) \quad \sum_{x \in X(a) \cap X(b)} T_x \geq (2^{t-s} - 1)|X(a) \cap X(b)|.$$

Now this sum counts the number of pairs  $(i, x)$  with  $2 \leq i \leq 2^t$ ,  $x \in X(a) \cap X(b)$  and  $-x \in D(\rho'_i)$ . We can also count the number of such pairs by first fixing  $i$ . Namely:

$$(11) \quad \sum_{x \in X(a) \cap X(b)} T_x = \sum_{i=2}^{2^t} |D(\rho'_i) \cap -(X(a) \cap X(b))|.$$

Now (11) implies that there exists an  $i$ ,  $2 \leq i \leq 2^t$ , such that:

$$|D(\rho'_i) \cap -(X(a) \cap X(b))| \geq \frac{1}{2^t - 1} \sum_{x \in X(a) \cap X(b)} T_x$$

and hence when combined with (9):

$$|D(\rho'_i)| \geq \frac{2^{t-s} - 1}{2^t - 1} |X(a) \cap X(b)|.$$

Applying equations (5) and (3) yields:

$$(12) \quad k^2 > \frac{2^{t-s} - 1}{2^t - 1} (g - 2k^2).$$

If  $s = 0$ , then (12) becomes  $k^2 > g - 2k^2$  which is impossible as  $g \geq 8k^2$  [3, 4.4]. Suppose then that  $s \geq 1$ . (12) is then:

$$(2^t + 2^{t-s+1} - 3)k^2 > (2^{t-s} - 1)g.$$

Use  $g \geq 2^{s+2}k^2$  from the first part of (8) to get:

$$\begin{aligned} (2^t + 2^{t-s+1} - 3)k^2 &> (2^{t+2} - 2^{s+2})k^2, \\ 2^{s+2} + 2^{t-s+1} - 3 &> 3 \cdot 2^t. \end{aligned}$$

Lastly, using  $t - 1 \geq s$  from the second part of (8) gives:

$$\begin{aligned} 2^{t+1} + 2^{t-s+1} - 3 &> 3 \cdot 2^t, \\ 2^{t-s+1} - 3 &> 2^t, \end{aligned}$$

which is impossible for  $s \geq 1$ . This contradiction shows  $g \geq k^3$ . □

We combine these results with Kula's upper bound on  $g$  and bound on  $k$ .

**Corollary 1.** *For an exceptional  $k$ -regular Witt ring  $(R, G)$  with  $g = |G|$ :  $k \geq 16$  and*

(1) *if  $k \equiv 1 \pmod{3}$ , then  $g \equiv 1 \pmod{3}$  and*

$$k^3 \leq g \leq \frac{1}{4}k^4,$$

(2) *if  $k \equiv 2 \pmod{3}$ , then*

$$k^3 \leq g \leq \frac{1}{8}k^4. \quad \square$$

We note that the first open case is  $k = 16$  and  $g = 16^3 = 4096$ .

Kula has shown that an exceptional  $k$ -regular Witt ring is non-formally real [3, Remark, p. 41] so that  $I^n R = 0$  for some  $n$ . We have:

**Corollary 2.** *If  $(R, G)$  is an exceptional  $k$ -regular Witt ring, then  $I^3 R \neq 0$ . In fact, for any anisotropic 2-fold Pfister form  $\rho$ ,  $D(\rho) \neq G$ .*

*Proof.*  $D(\rho) = \bigcup_{b \in D(\rho')} D(1, b)$  so that  $|D(\rho)| \leq k|D(\rho')| < k^3$  by equation (5). Thus  $D(\rho) \neq G$  by Theorem 1. □

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