**K-REGULAR WITT RINGS**

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(Communicated by Lance W. Small)

**Abstract.** We improve Kula’s bounds on the size of possible $k$-regular Witt rings.

$(R, G, q)$ will denote an abstract Witt ring in the sense of [4]. Nearly all examples of interest are Witt rings of non-singular quadratic forms over a field of characteristic not two; however, using abstract Witt rings does simplify some proofs. The Witt ring is $k$-regular if there exists a 2-power $k$ such that for all $1 \neq x \in G$ we have $|D(1, -x)| = k$. Such Witt rings were first studied in [1] primarily because the block design counting arguments there were perfectly suited to $k$-regular rings. However they remain unclassified.

We will always assume that $G$ is finite; set $g = |G|$. If $k = g$, then $R$ is totally degenerate and so classified by [4]. If $k = g/2$, then $R$ is of local type [2] which is again classified in [4]. If $k = 2$, then $R$ is a group ring extension of $\mathbb{Z}_2$ or $\mathbb{Z}_4$. If $2 < k < g/2$, then $R$ is not of elementary type and no examples are known or even expected. We will always assume that $2 < k < g/2$ and call such $k$-regular Witt rings exceptional.

It was shown in [1] that exceptional $k$-regular Witt rings satisfy $8 \leq k$ and $2k^2 \leq g$. Kula [3] improved both bounds and added an upper bound, showing:

$$16 \leq k,$$

$$8k^2 \leq g \leq k^4/4 \quad \text{if } k \equiv 1 \pmod{3},$$

$$8k^2 \leq g \leq k^4/8 \quad \text{if } k \equiv 2 \pmod{3}.$$

Here we show that $k^4 \leq g$ and that if $k \equiv 1 \pmod{3}$ then $g \equiv 1 \pmod{3}$.

We fix some notation, which will agree with Kula’s. $G^*$ denotes $G \setminus \{1\}$. We set $e = \log_2 k$. For $a \in G^*$ and $i \geq 0$ set:

$$X_i(a) = \{x \in G: x \neq 1, a \text{ and } |Q(a) \cap Q(x)| = 2^i\},$$

where $Q(x) = \{q(x, y): y \in G\}$. Now for $x \neq a$,

$$|Q(a) \cap Q(x)| = |D(1, -ax)|/|D(1, -a) \cap D(1, -x)| = k/|D(1, -a) \cap D(1, -x)|.$$

Thus we also have that:

$$X_i(a) = \{x \in G: x \neq 1, a \text{ and } |D(1, -a) \cap D(1, -x)| = 2^{e-i}\}.$$
In particular, we may assume $0 \leq i \leq e$. We further set $n_i(a) = |X_i(a)|$ and write $X(a)$ for $X_e(a)$. For a 2-fold Pfister form $\rho$ we let $\rho'$ denote the pure part of $\rho$.

We will use various equations derived by Kula:

(1) \[ \sum_{i=0}^{e-1} (2^{e-i} - 1) n_i(a) = k^2 - 3k + 2, \]

(2) \[ g + \sum_{1 \neq \rho \in Q(a)} |D(\rho')| = 1 + \frac{g}{k} + \sum_{i=0}^{e} 2^i n_i(a), \]

(3) \[ |X(a) \cap X(b)| \geq g - 2k^2 + 6k - 7 \geq g - 2k^2, \]

where $a \neq b$ in $G^*$ for (3). Equation (1) is [3, 4.3b], (2) is equation (4.5.2) in [3, p. 45] and the first inequality of (3) is equation (4.3.1) in [3, p. 43]. The second inequality of (3) follows from our assumption that $k > 2$.

We will also use two simple equations:

(4) \[ \sum_{i=0}^{e} n_i(a) = g - 2, \]

(5) \[ |D(\rho')| < k^2 \quad \text{(if } \rho \neq 1). \]

Both (4) and (5) appear in [3] but direct proofs are quick. (4) follows from $G \setminus \{1, a\}$ being the union of the $X_i(a)$. For (5), suppose $\rho' = (a, b, ab)$. Then

\[ D(\rho') = a \cdot \bigcup_{x \in D(1, a)} D(1, bx). \]

Since 1 occurs in each $D(1, bx)$ we have that $|D(\rho')| < |D(1, a)| \cdot k = k^2$.

Using equation (4) to find $n_e(a)$ and equation (1) to find $n_{e-1}(a)$, equation (2) may be re-written (see [3, pp. 45–46]) as:

\[ g + \sum_{1 \neq \rho \in Q(a)} |D(\rho')| = 1 + \frac{g}{k} + gk - \frac{k^3}{2} + \frac{3k^2}{2} - 3k \]

\[ + \sum_{i=0}^{e-2} 2^i (2^{e-i-1} - 1)(2^{e-i} - 1)n_i(a). \]

(6)

**Proposition 1.** If $k \equiv 1 \pmod{3}$, then $g \equiv 1 \pmod{3}$.

**Proof.** We may pick an $a \in G^*$ with $\langle (1, 1) \rangle \notin Q(a) \setminus \{1\}$ (otherwise $-G^* \subset D(1, 1, 1)$ while $|D(1, 1, 1)| < k^2$ by (5) and $|G^*| \geq 8k^2 - 1$ by [3, 4.4]). Then for each anisotropic $\rho \in Q(a)$ we have that $|D(\rho')| \equiv 0 \pmod{3}$ by [3, 2.9]. Also, since for each $i$, in equation (6) one of $e - i - 1$ or $e - i$ is even, we have that one of $2^{e-i-1} - 1$ or $2^{e-i} - 1$ is divisible by 3. Assuming $k \equiv 1 \pmod{3}$, equation (6) gives:

\[ g \equiv g + 1 + g - 2 \pmod{3}, \]

and so $1 \equiv g \pmod{3}$.

**Theorem 1.** $g \geq k^3$. 

\[ \square \]
Proof. Suppose there exists an exceptional $k$-regular Witt ring $(R, G)$ with $g < k^3$. Among all such Witt rings, choose one with minimal $h = g/k^2$. Let $a$ and $b$ be distinct elements of $G^*$. Choose $x \in X(a) \cap X(b)$, which is possible by equation (3) and the fact that $g \geq 8k^2$ [3, 4.4]. We use the equation (4.3.2) from [3, p. 43]:

$$hk = g/k = |Q(x)| \geq |(Q(x) \cap Q(a))(Q(x) \cap Q(b))| = \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} \geq \frac{k^2}{|Q(a) \cap Q(b)|} = k|D(1,-a) \cap D(1,-b)|.$$  

(7)

A simple consequence of (7) is that $|D(1,-a) \cap D(1,-b)| \leq h$. Pick a minimal $s \geq 0$ so that there exist distinct $a$ and $b$ in $G^*$ with $|D(1,-a) \cap D(1,-b)| = h/2^s$. Set $2^t = |Q(a) \cap Q(b)|$. Then we have:

$$g \geq 2^{s+2}k^2 \quad \text{and} \quad t - s \geq 1.$$  

(8)

Namely, if the first inequality failed, then $h = g/k^2 \leq 2^{s+1}$. But then $|D(1,-a) \cap D(1,-b)| \leq 2$ for all distinct $a$ and $b$ in $G^*$, while as noted in the first sentence of [3, p. 44] we can always find distinct $a$ and $b$ in $G^*$ with $|D(1,-a) \cap D(1,-b)| \geq 4$. For the second inequality of (8) note that:

$$2^t = |Q(a) \cap Q(b)| = \frac{k}{|D(1,-a) \cap D(1,-b)|} = \frac{2^sk}{h}.$$  

Thus $2^{t-s}h = k$. By the assumption that $g = hk^2 < k^3$ we have $2h \leq k$ and so $t - s \geq 1$.

For each $x \in X(a) \cap X(b)$ we can rewrite (7) as:

$$hk = \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} \geq \frac{k^2}{|Q(a) \cap Q(b)|} = \frac{hk}{2^s}.$$  

(9)

Then

$$|Q(x) \cap Q(a) \cap Q(b)| \geq 2^{t-s}$$

since otherwise $|Q(x) \cap Q(a) \cap Q(b)| < 2^{t-s} = |Q(a) \cap Q(b)|/2^s$ and equation (9) becomes:

$$hk \geq \frac{k^2}{|Q(x) \cap Q(a) \cap Q(b)|} > \frac{2^sk^2}{|Q(a) \cap Q(b)|} = hk.$$  

List the elements of $Q(a) \cap Q(b)$ as $1, \rho_2, \ldots, \rho_{2^t}$. We have for each $x \in X(a) \cap X(b)$ that $2^{t-s} - 1$ of the $\rho_i$’s lie in $Q(x)$, or equivalently, satisfy $-x \in D(\rho_i)$. Set $T_x$ equal to the number of $i$’s, $2 \leq i \leq 2^t$, such that $-x \in D(\rho_i)$. Then:

$$\sum_{x \in X(a) \cap X(b)} T_x \geq (2^{t-s} - 1)|X(a) \cap X(b)|.$$  

(10)

Now this sum counts the number of pairs $(i, x)$ with $2 \leq i \leq 2^t$, $x \in X(a) \cap X(b)$ and $-x \in D(\rho_i)$. We can also count the number of such pairs by first fixing $i$. Namely:

$$\sum_{x \in X(a) \cap X(b)} T_x = \sum_{i=2}^{2^t} |D(\rho_i) \cap -(X(a) \cap X(b))|.$$  

(11)
Now (11) implies that there exists an \( i, 2 \leq i \leq 2t \), such that:

\[
|D(\rho_i') \cap -(X(a) \cap X(b))| \geq \frac{1}{2t - 1} \sum_{x \in X(a) \cap X(b)} T_x
\]

and hence when combined with (9):

\[
|D(\rho_i')| \geq \frac{2^{t-s}-1}{2t-1} |X(a) \cap X(b)|.
\]

Applying equations (5) and (3) yields:

\[
k^2 > \frac{2^{t-s}-1}{2t-1} (g - 2k^2).
\]

If \( s = 0 \), then (12) becomes \( k^2 > g - 2k^2 \) which is impossible as \( g \geq 8k^2 \) [3, 4.4]. Suppose then that \( s \geq 1 \). (12) is then:

\[
(2^t + 2^{t-s+1} - 3)k^2 > (2^{t-s} - 1)g.
\]

Use \( g \geq 2s+2k^2 \) from the first part of (8) to get:

\[
(2^t + 2^{t-s+1} - 3)k^2 > (2^{t+s} - 2^{s+2})k^2,
\]

\[
2^{s+2} + 2^{t-s+1} - 3 > 3 \cdot 2^t.
\]

Lastly, using \( t - 1 \geq s \) from the second part of (8) gives:

\[
2^{t+1} + 2^{t-s+1} - 3 > 3 \cdot 2^t,
\]

\[
2^{t-s+1} - 3 > 2^t,
\]

which is impossible for \( s \geq 1 \). This contradiction shows \( g \geq k^3 \).

We combine these results with Kula’s upper bound on \( g \) and bound on \( k \).

**Corollary 1.** For an exceptional \( k \)-regular Witt ring \((R, G)\) with \( g = |G| \): \( k \geq 16 \) and

1. if \( k \equiv 1 \pmod{3} \), then \( g \equiv 1 \pmod{3} \) and

\[
k^3 \leq g \leq \frac{1}{4}k^4,
\]

2. if \( k \equiv 2 \pmod{3} \), then

\[
k^3 \leq g \leq \frac{1}{5}k^4.
\]

We note that the first open case is \( k = 16 \) and \( g = 16^3 = 4096 \).

Kula has shown that an exceptional \( k \)-regular Witt ring is non-formally real [3, Remark, p. 41] so that \( T^n R = 0 \) for some \( n \). We have:

**Corollary 2.** If \((R, G)\) is an exceptional \( k \)-regular Witt ring, then \( T^n R \neq 0 \). In fact, for any anisotropic 2-fold Pfister form \( \rho \), \( D(\rho) \neq G \).

Proof. \( D(\rho) = \bigcup_{b \in D(\rho')} D(1, b) \) so that \( |D(\rho)| \leq k|D(\rho')| < k^3 \) by equation (5). Thus \( D(\rho) \neq G \) by Theorem 1.
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REFERENCES


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