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THE EXPECTED VALUE OF THE NUMBER OF REAL ZEROS OF A RANDOM SUM OF LEGENDRE POLYNOMIALS

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ABSTRACT. It is known that the expected number of zeros in the interval $(-1,1)$ of the sum $a_0\psi_0(t)+a_1\psi_1(t)+\cdots+a_n\psi_n(t)$, in which $\psi_k(t)$ is the normalized Legendre polynomial of degree k and the coefficients a_k are independent normally distributed random variables with mean 0 and variance 1, is asymptotic to $3^{-1/2}n$ for large n. We improve this result and show that this expected number is $3^{-1/2}n+o(n^\delta)$ for any positive δ.

1. Introduction

Let k be a nonnegative integer, $P_k(t)$ be the Legendre polynomial of degree k, and $\psi_k(t)$ be the normalised Legendre polynomial $(k+\frac{1}{2})^{\frac{1}{2}}P_k(t)$. Let n be a positive integer, and consider the random sum

$$F(t)=\sum_{k=0}^{n}a_k\psi_k(t),$$

in which the coefficients a_k are independent normally distributed random variables with mean 0 and variance 1. If ν_n is the expected value of the number of zeros of $F(t)$ on the interval $(-1,1)$, then Das [1] has shown that $\nu_n\sim 3^{-\frac{1}{2}}n$ for large n. (In fact, his analysis indicates that $\nu_n=3^{-\frac{1}{2}}n[1+O\{(\log n)^{-3}\}]$.) In this paper we will prove the somewhat better result that

$$\nu_n=3^{-\frac{1}{2}}n+o(n^\delta) \tag{1}$$

for any positive δ. Our analysis is similar to that of Das, but requires a more detailed treatment of the asymptotic expansion for $P_n(t)$ when n is large.

2. Preliminary analysis

Let $\nu_n(a,b)$ be the expected number of zeros of $F(t)$ on the subinterval (a,b) of $(-1,1)$. We know ([1] or [2, p. 111]) that

$$\nu_n(a,b)=\pi^{-1}\int_a^b\{A_n(t)C_n(t)-B_n^2(t)\}^{\frac{1}{2}}A_n^{-1}(t)\,dt, \tag{2}$$

$$A_n(t)=P'_{n+1}(t)P_n(t)-P'_n(t)P_{n+1}(t), \tag{3}$$

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$$2B_n(t) = P''_{n+1}(t)P_n(t) - P''_n(t)P_{n+1}(t), \tag{4}$$

$$6C_n(t) = 3\{P''_{n+1}(t)P'_n(t) - P''_n(t)P'_{n+1}(t)\} + \{P'''_{n+1}(t)P_n(t) - P'''_n(t)P_{n+1}(t)\}. \tag{5}$$

Stieltjes [3, 4] (or see [5, p. 195, Th. 8.21.5]) has shown that, if $0 < \alpha < \pi$ and

$$(n \sin\alpha)^{-1} = o(1), \tag{6}$$

then

$$P_n(\cos\alpha) = \left\{\frac{2}{(\pi\sin\alpha)}\right\}^{\frac{1}{2}} \left[\sum_{s=0}^{m-1} \frac{\gamma_s n! \cos\beta_{ns}}{(2\sin\alpha)^s \Gamma(n+s+3/2)} + O\{(n\sin\alpha)^{-m}\}\right], \tag{7}$$

in which $\gamma_s = \{\Gamma(s+\frac{1}{2})\}^2/(\pi s!)$, $\beta_{ns} = \beta_{ns}(\alpha) = (n+s+\frac{1}{2})\alpha - \frac{1}{2}(s+\frac{1}{2})\pi$, and m is any positive integer. Moreover, the upper bound implicit in the O symbol depends only on m, not only in (7), but in the later identities (9), (10), (11), (14), (16), (18), (21), (22), (23) and (24). If we define $G_{kn}(\alpha)$ so that

$$G_{kn}(\alpha) = \sum_{s=0}^{k} \gamma_s D_{k-s}(1, s+3/2)(2\sin\alpha)^{k-s}\cos\beta_{ns}, \tag{8}$$

in which the coefficients $D_h(x,y)$ are those that appear in the asymptotic expansion [6, p. 119, Eq. 5.02]

$$n^{y-x}\Gamma(n+x)/\Gamma(n+y) = \sum_{h=0}^{p-1} D_h(x,y)n^{-h} + O(n^{-p}),$$

then $G_{kn} = G_{kn}(\alpha)$ is uniformly bounded in n and α, and we deduce from (7) that

$$P_n(\cos\alpha) = \left\{\frac{2}{(n\pi\sin\alpha)^{\frac{1}{2}}}\right\} \left[\sum_{k=0}^{m-1} (2n\sin\alpha)^{-k} G_{kn} + O(n^{-m}\sin^{-m}\alpha)\right]. \tag{9}$$

Because $D_0(x,y) = 1$, we see that $G_{on}(\alpha) = \cos\beta_{no}$. Moreover,

$$P_{n+1}(\cos\alpha) = \left\{\frac{2}{(n\pi\sin\alpha)}\right\}^{\frac{1}{2}} \left[\sum_{k=0}^{m-1} (2n\sin\alpha)^{-k} H_{kn} + O(n^{-m}\sin^{-m}\alpha)\right], \tag{10}$$

$$P_{n-1}(\cos\alpha) = \left\{\frac{2}{(n\pi\sin\alpha)}\right\}^{\frac{1}{2}} \left[\sum_{k=0}^{m-1} (2n\sin\alpha)^{-k} J_{kn} + O(n^{-m}\sin^{-m}\alpha)\right], \tag{11}$$

$$H_{kn} = \sum_{s=0}^{k} {}_{-s-\frac{1}{2}}C_{k-s}(2\sin\alpha)^{k-s} G_{s,n+1}, \tag{12}$$

$$J_{kn} = \sum_{s=0}^{k} {}_{-s-\frac{1}{2}}C_{k-s}(-2\sin\alpha)^{k-s} G_{s,n-1}, \tag{13}$$

in which ${}_qC_j = \Gamma(j+1)/\{\Gamma(q+1)\Gamma(j-q+1)\}$. The functions H_{kn} and J_{kn} are uniformly bounded in n and α, and $H_{on} = \cos\beta_{n+1,o}$, $J_{on} = \cos\beta_{n-1,o}$.

We use the identity [7, p. 309, eq. V]

$$(1-t^2)P'_n(t) = n\{P_{n-1}(t) - tP_n(t)\}$$

in conjunction with (9) and (11) to see that

$$P'_n(\cos\alpha) = \left\{\frac{2n}{(\pi\sin^3\alpha)}\right\}^{\frac{1}{2}} \left[\sum_{k=0}^{m-1}(2n\sin\alpha)^{-k}K_{kn} + O(n^{-m}\sin^{-m-1}\alpha)\right], \tag{14}$$

$$K_{kn} = (J_{kn} - G_{kn}\cos\alpha)/\sin\alpha. \tag{15}$$

It is now convenient to define a congruence relation $\equiv$ between two functions X and Y of n and α so that $X \equiv Y$ in case $(X-Y)/\sin\alpha$ is uniformly bounded in n and α. This relation is an equivalence relation that is preserved under addition and under multiplication by uniformly bounded functions. It follows from (13) that $J_{kn} \equiv G_{k,n-1}$ and from (8) that $G_{kn} \equiv \gamma_k \cos\beta_{nk}$. Hence $J_{nk} - G_{nk}\cos\alpha \equiv \gamma_k(\cos\beta_{n-1,k} - \cos\beta_{nk}\cos\alpha) = \gamma_k \sin\beta_{nk}\sin\alpha$, so that the function K_{kn} defined in (15) is uniformly bounded in n and α. In particular, $K_{on} = \sin\beta_{no}$. Moreover,

$$P'_{n+1}(\cos\alpha) = \left\{\frac{2n}{(\pi\sin^3\alpha)}\right\}^{\frac{1}{2}} \left[\sum_{k=0}^{m-1}(2n\sin\alpha)^{-k}L_{kn} + O(n^{-m}\sin^{-m-1}\alpha)\right], \tag{16}$$

$$L_{kn} = \sum_{s=0}^{k} {}_{\frac{1}{2}-s}C_{k-s}(2\sin\alpha)^{k-s}K_{s,n+1}. \tag{17}$$

Therefore, L_{kn} is uniformly bounded in n and α, and $L_{on} = \sin\beta_{n+1,o}$.

It now follows from (3), (9), (10), (14) and (16) that, if $t = \cos\alpha$,

$$A_n(t) = \left\{\frac{2n}{(\pi\sin\alpha)}\right\} \left[\sum_{k=0}^{m-1}(2n\sin\alpha)^{-k}M_{kn} + O(n^{-m}\sin^{-m-2}\alpha)\right], \tag{18}$$

$$M_{kn} = \sum_{s=0}^{k}(L_{sn}G_{k-s,n} - K_{sn}H_{k-s,n})/\sin\alpha. \tag{19}$$

We deduce from (17), (8) and (12) that

$$\begin{aligned} L_{sn}G_{k-s,n} - K_{sn}H_{k-s,n} &\equiv \gamma_{k-s}(K_{s,n+1}\cos\beta_{n,k-s} - K_{sn}\cos\beta_{n+1,k-s}) \\ &\equiv \gamma_{k-s}(K_{s,n+1} - K_{sn}\cos\alpha)\cos\beta_{n,k-s}. \end{aligned}$$

We infer from (15), (13) and (8) that $K_{sn} = U'_{sn} + U''_{sn}$, in which

$$\begin{aligned} U'_{sn} &= (J_{sn} - G_{s,n-1})/\sin\alpha = \sum_{k=0}^{s-1} {}_{-k-\frac{1}{2}}C_{s-k}(-2\sin\alpha)^{s-1-k}G_{k,n-1} \\ &\equiv -\left(s-\tfrac{1}{2}\right)G_{s-1,n-1} \equiv -\left(s-\tfrac{1}{2}\right)\cos\beta_{n-1,s-1}, \\ U''_{sn} &= (G_{s,n-1} - G_{sn}\cos\alpha)/\sin\alpha \\ &= \sum_{k=0}^{s}\gamma_r D_{s-k}(1,k+3/2)(2\sin\alpha)^{s-k}\sin\beta_{nk} \equiv \gamma_s\sin\beta_{ns}. \end{aligned}$$

We next observe that

$$\begin{aligned} U'_{s,n+1} - U'_{sn}\cos\alpha &\equiv -(s-\tfrac{1}{2})(\cos\beta_{n,s-1} - \cos\beta_{n-1,s-1}\cos\alpha) \\ &= (s-\tfrac{1}{2})\sin\beta_{n-1,s-1}\sin\alpha \equiv 0, \\ U''_{s,n+1} - U''_{sn}\cos\alpha &\equiv \gamma_s(\sin\beta_{n+1,s} - \sin\beta_{ns}\cos\alpha) \\ &= \gamma_s\cos\beta_{ns}\sin\alpha \equiv 0. \end{aligned}$$

It follows from (19) that M_{kn} is uniformly bounded in n and α, and $M_{on} = 1$.

It is a consequence of the differential equation [7, p. 304]

$$(1-t^2)P_n''(t) - 2tP_n'(t) + n(n+1)P_n(t) = 0, \tag{20}$$

satisfied by $P_n(t)$, and the definitions (3) and (4) that

$$(1-t^2)B_n(t) = tA_n(t) - (n+1)P_n(t)P_{n+1}(t).$$

It then follows from (18), (9) and (10) that

$$B_n(t) = \left\{\frac{2}{(\pi \sin^3 \alpha)}\right\} \left[\sum_{k=0}^{m-1} (2n \sin\alpha)^{-k} N_{kn} + O(n^{-m} \sin^{-m-2}\alpha)\right], \tag{21}$$

in which N_{kn} is a function, uniformly bounded in n and α, whose explicit expression is not needed.

An additional consequence of (20) is that (it is convenient to suppress the dependence on t of the Legendre polynomials and their derivatives)

$$(1-t^2)(P_{n+1}''P_n' - P_n''P_{n+1}') = (n+1)(nA_n - 2P_{n+1}P_n').$$

If we differentiate (20) and use the definitions (3) and (4), we find that

$$(1-t^2)(P_{n+1}'''P_n - P_n'''P_{n+1}) = n^2A_n + 8tB_n + (n-2)P_{n+1}P_n' - 3nP_{n+1}'P_n.$$

It then follows from (5) and the last two equations that

$$6(1-t^2)C_n = (2n^2+3n)A_n + 8tB_n - (5n+8)P_{n+1}P_n' - 3nP_{n+1}'P_n.$$

An appeal to (18), (21), (10), (14), (16) and (7) shows that

$$C_n(t) = \left\{\frac{2n^2}{(3\pi \sin^3 \alpha)}\right\} \left[\sum_{k=0}^{m-1} (2n \sin\alpha)^{-k} Q_{kn} + O(n^{-m} \sin^{-m-2}\alpha)\right], \tag{22}$$

in which Q_{kn} is a function, uniformly bounded in n and α, whose explicit expression is not needed, except for the case $Q_{on} = 1$.

With the help of (18), (21) and (22), we now find that

$$\begin{aligned} &\{A_n(t)C_n(t) - B_n^2(t)\}^{\frac{1}{2}} A_n^{-1}(t) \\ &\quad = \left(\frac{3^{-\frac{1}{2}}n}{\sin\alpha}\right) \left[\sum_{k=0}^{m-1} (2n \sin\alpha)^{-k} R_{kn} + O(n^{-m} \sin^{-m-2}\alpha)\right], \end{aligned} \tag{23}$$

in which R_{kn} is a function, uniformly bounded in n and α, whose explicit expression is not needed, except for the case $R_{on} = 1$.

3. Proof of (1)

Suppose that $\varepsilon = n^{-2m/(m+4)}$, and that $|t| \le 1-\varepsilon$. Then $n\sin\alpha = n(1-t^2)^{\frac{1}{2}} > n\varepsilon^{\frac{1}{2}} = n^{4/(m+4)}$. Hence (6) is true. We conclude from (2) and (23) that

$$\begin{aligned} \nu_n(-1+\varepsilon, 1-\varepsilon) &= 2\nu_n(0, 1-\varepsilon) \\ &= \left\{\frac{2n}{(3^{\frac{1}{2}}\pi)}\right\} \int_0^{1-\varepsilon} [(1-t^2)^{-\frac{1}{2}} + \sum_{k=1}^{m-1} O\{n^{-k}(1-t^2)^{-(k+1)/2}\} \\ &\qquad + O\{n^{-m}(1-t^2)^{-(m+5)/2}\}]\, dt. \end{aligned}$$

We observe that

$$\int_0^{1-\varepsilon}(1-t^2)^{-\frac{1}{2}}dt = (\pi/2) - \cos^{-1}(1-\varepsilon) = (\pi/2) + O(\varepsilon^{\frac{1}{2}}),$$

$$\int_0^{1-\varepsilon}(1-t^2)^{-1}dt = \tfrac{1}{2}\log\{(2-\varepsilon)/\varepsilon\} = O(\log\varepsilon^{-1}),$$

$$\int_0^{1-\varepsilon}(1-t^2)^{-3/2}dt = (1-\varepsilon)(2\varepsilon-\varepsilon^2)^{-\frac{1}{2}} = O(\varepsilon^{-\frac{1}{2}}).$$

Because $1-t^2 > \varepsilon$ when $0 \le t \le 1-\varepsilon$, we see that

$$\int_0^{1-\varepsilon}(1-t^2)^{(-h-\frac{1}{2})}dt < \varepsilon^{(1-h)}\int_0^{1-\varepsilon}(1-t^2)^{-3/2}dt = O(\varepsilon^{-h+\frac{1}{2}})$$

when $h \ge 1$. Therefore,

$$\begin{aligned}\nu_n(-1+\varepsilon, 1-\varepsilon) =& 3^{-\frac{1}{2}}n[1 + O(\varepsilon^{\frac{1}{2}}) + O(n^{-1}\log\varepsilon^{-1}) \\ &+ \varepsilon^{\frac{1}{2}}\sum_{k=2}^{m-1}O\{(n\varepsilon^{\frac{1}{2}})^{-k}\} + O(n^{-m}\varepsilon^{-(m+3)/2})],\end{aligned}$$

$$\nu_n(-1+\varepsilon, 1-\varepsilon) = 3^{-\frac{1}{2}}n[1 + O\{n^{-m/(m+4)}\}]. \tag{24}$$

Let $\mu(\varepsilon)$ be the number of complex zeros of $F(t)$ in the circle $|t-1| < \varepsilon$. Then $\nu_n(1-\varepsilon, 1)$ does not exceed the expected value of $\mu(\varepsilon)$. It follows from Jensen's Theorem [8, p. 187, Eq. 25]

$$\mu(\varepsilon) \le (2\pi)^{-1}\int_0^{2\pi}\log_2|F(1+2\varepsilon e^{i\theta})/F(1)|d\theta. \tag{25}$$

We next use the identity [7, p. 312],

$$P_k(z) = \pi^{-1}\int_0^{\pi}\{z + (z^2-1)^{\frac{1}{2}}\cos\varphi\}^k\,d\varphi,$$

to see that

$$|P_k(1+2\varepsilon e^{i\theta})| \le \{1 + 2\varepsilon + 2(\varepsilon+\varepsilon^2)^{\frac{1}{2}}\}^k < (1 + A\varepsilon^{\frac{1}{2}})^k,$$

in which $A = 2(1+2^{\frac{1}{2}}) < 5$. Therefore, when $0 \le k \le n$,

$$|P_k(1+2\varepsilon e^{i\theta})| < \exp\{n\log(1+5\varepsilon^{\frac{1}{2}})\} < \exp(5n\varepsilon^{\frac{1}{2}}). \tag{26}$$

The Chebyshev inequality [9, p. 219, Eq. 61] shows that $Prob(|a_k| \le n) > 1 - n^{-2}$ for each k, such that

$$Prob(|a_k| \le n \text{ when } 0 \le k \le n) > 1 - (n+1)n^{-2} \ge 1 - 2n^{-1}. \tag{27}$$

Because the Schwarz inequality implies that

$$\sum_{k=0}^{n}\left(k+\frac{1}{2}\right)^{\frac{1}{2}} \le \left\{(n+1)\sum_{k=0}^{n}\left(k+\frac{1}{2}\right)\right\}^{\frac{1}{2}} = \{(n+1)^3/2\}^{\frac{1}{2}} < n^{3/2}$$

when $n \ge 4$, it follows from (26) and (27) that

$$Prob\{|F(1+2\varepsilon e^{i\theta})| < n^{5/2}\exp(5n\varepsilon^{\frac{1}{2}})\} > 1 - 2n^{-1}. \tag{28}$$

Moreover, $F(1)$ is a normally distributed random variable with mean 0 and variance $\zeta^2 = (n+1)^2/2$. Therefore,

(29)

$$Prob\{|F(1)| < 1\} = (2\pi\zeta^2)^{-\frac{1}{2}} \int_{-1}^{1} \exp(-u^2/2\zeta^2) du < 2\pi^{-\frac{1}{2}}/(n+1) < 2n^{-1}.$$

We infer from (25), (28) and (29) that $\mu(\varepsilon) < \log_2\{n^{5/2}\exp(5n\varepsilon^{\frac{1}{2}})\}$ with probability greater than $1-4n^{-1}$. Because $\mu(\varepsilon) \leq n$ for all $F(t)$, we see that the expected value of $\mu(\varepsilon)$ is, and so also $\nu_n(1-\varepsilon, 1)$ and $\nu_n(-1, -1+\varepsilon)$ are, $O(\log n) + O(n\varepsilon^{\frac{1}{2}}) + O(1) = O(n^{4/(m+4)})$. When this result is combined with (24), we find that $\nu_n = \nu_n(-1, 1) = 3^{-\frac{1}{2}}n + O\{n^{4/(m+4)}\}$. If δ is any positive number and we choose the integer m so large that $m+4 > 4/\delta$, we can finally conclude that (1) is true, in the sense that for any positive η and δ there exists an integer $n(\eta, \delta)$ such that $|\nu_n - 3^{-\frac{1}{2}}n|n^{-\delta} < \eta$ when $n > n(\eta, \delta)$.

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