A NEW PROOF OF THE TWO WEIGHT NORM INEQUALITY FOR THE ONE-SIDED FRACTIONAL MAXIMAL OPERATOR

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(Communicated by J. Marshall Ash)

Abstract. We give a new proof of the two weight norm inequality for the one-sided, fractional maximal operator, simplifying the original proof of Martín-Reyes and de la Torre.

1. Introduction

In [1], Andersen and Sawyer introduced the one-sided fractional maximal operators

\[ M^+_\alpha f(x) = \sup_{t > 0} \frac{1}{t^{1-\alpha}} \int_{x+t}^{x} |f| \, dy \quad \text{and} \quad M^-_\alpha f(x) = \sup_{t > 0} \frac{1}{t^{1-\alpha}} \int_{x-t}^{x} |f| \, dy, \]

0 < \alpha < 1, in order to study the weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators. Using complex interpolation, they proved one-weight norm inequalities for \( M^+_{\alpha} \) and \( M^-_{\alpha} \). In [3], Martín-Reyes and de la Torre answered a question of Andersen and Sawyer by giving a geometric proof of a two-weight norm inequality for \( M^+_{\alpha} \). (The same result holds, \textit{mutatis mutandis}, for \( M^-_{\alpha} \).)

Theorem 1.1. For non-negative weights \( u \) and \( v \), and for \( 1 < p \leq q \), the following are equivalent:

1. There exists a constant \( C \) such that, for every function \( f \) in \( L^p(v) \),

\[ \left( \int_{\mathbb{R}} (M^+_{\alpha} f)^q u \, dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}} |f|^p v \, dx \right)^{1/p}; \]

2. The pair \((u, v)\) satisfy the \( (S^+_{p,q,\alpha}) \) condition: there exists a constant \( C \) such that for every interval \( I = [a,b] \) for which \( u((-\infty, a)) > 0 \),

\[ \left( \int_I (M^+_{\alpha} (\sigma \chi_I))^q u \, dx \right)^{1/q} \leq C \left( \int_I \sigma \, dx \right)^{1/p} < \infty, \]

where \( \sigma = v^{1-p'} \).

Received by the editors August 29, 1995 and, in revised form, October 25, 1995 and November 16, 1995.

1991 Mathematics Subject Classification. Primary 42B25.

Key words and phrases. One-sided fractional maximal operator, weighted norm inequalities.
Their proof involved proving a weighted norm inequality for a dyadic variant of the fractional maximal operator, $M_{\alpha}^f$, using a dyadic version of the Sawyer condition, $(S_{p,\alpha}^+ D)$, and then showing that these were equivalent to $M_{\alpha}^+$ and $(S_{p,\alpha}^+)$. The purpose of this paper is to give a new proof of Theorem 1.1, one which eliminates the dyadic maximal operator. We do this by adapting the proof in the dyadic case using special covering properties of $\mathbb{R}$ and the continuity properties of the maximal operator. We believe that these techniques will be useful in proving other norm inequalities for maximal operators on $\mathbb{R}$.

The paper is organized as follows: Section 2 contains three lemmas and some remarks on their applicability, and Section 3 contains the actual proof. Throughout, all functions are assumed to be measurable, $C$ denotes a positive constant whose value may be different at each appearance, $p' = p/(p-1)$ is the conjugate exponent of $p$, and $0 < \alpha < 1$. Given a Borel set $E$ and a function $w$, let $|E|$ denote the Lebesgue measure of $E$ and $w(E) = \int_E w \, dx$.

2. Preliminary results

Lemma 2.1. Let $f$ be a non-negative, bounded, upper semicontinuous function of compact support. Then $M_{\alpha}^+ f$ is continuous.

Proof. Since $M_{\alpha}^+ f$ is always lower semicontinuous, it will suffice to show that it is upper semicontinuous at each point $x \in \mathbb{R}$. We will show this by contradiction: suppose that there exist an $\epsilon > 0$, a point $x_0$ and a sequence of points $\{x_n\}$ converging to $x_0$ such that $M_{\alpha}^+ f(x_n) > M_{\alpha}^+ f(x_0) + \epsilon$. For each $n$ there exists an interval $J_n$ whose left endpoint is $x_n$ such that

$$\frac{1}{|J_n|^{1-\alpha}} \int_{J_n} |f| \, dx > M_{\alpha}^+ f(x_0) + \epsilon. \tag{1}$$

Let the support of $f$ be contained in the (finite) interval $I$, and let $K$ be a finite, open interval containing $I$ and the $x_n$’s, $n \geq 0$. Then we may assume that each $J_n$ is contained in $K$. Therefore, after passing to a subsequence we may assume that the $J_n$’s converge to a possibly degenerate interval $J$ whose left endpoint is $x_0$. If $|J| > 0$ then inequality (1) implies that

$$M_{\alpha}^+ f(x_0) \geq \frac{1}{|J|^{1-\alpha}} \int_J |f| \, dx \geq M_{\alpha}^+ f(x_0) + \epsilon,$$

a contradiction. If $J = \{x_0\}$ then for $n$ sufficiently large, by the upper semicontinuity of $f$, $f(x) - f(x_0) < \epsilon$ for all $x \in J_n$. Therefore,

$$\frac{1}{|J_n|^{1-\alpha}} \int_{J_n} f(x) \, dx < |J_n|^{\alpha} (f(x_0) + \epsilon) .$$

If we combine this with inequality (1) and take the limit as $n$ tends to infinity, we get (since $f$ is bounded) that $0 \geq M_{\alpha}^+ f(x_0) + \epsilon$, which is again a contradiction. Hence $M_{\alpha}^+ f$ is upper semicontinuous at each point, and we are done. 

It is worth noting that this lemma is not true if $\alpha = 0$. A simple counter-example is given by the characteristic function of $[0, 1]$. (I am grateful to Juha Kinnunen for pointing this out to me.)

The next lemma is due to Jesus Aldaz; the proof is in Bliedtner and Loeb [2].
Lemma 2.2. If \( \mu \) is a finite Borel measure on \( \mathbb{R} \), and if \( \mathcal{I} \) is an arbitrary collection of non-degenerate intervals, then for each \( \delta > 0 \) there exists a finite subcollection, \( \mathcal{I}_\delta \), of disjoint intervals in \( \mathcal{I} \) such that

\[
\mu\left( \bigcup_{I \in \mathcal{I}} I \right) \leq (2 + \delta) \sum_{I_k \in \mathcal{I}_\delta} \mu(I_k).
\]

Below we will want to apply Lemma 2.2 with the measure \( u \, dx \), where \((u,v)\) satisfies the \( (S_{p,q}^+) \) condition. To do this, we need \( u \) to be locally integrable and the intervals to be contained in some compact set. However, if \( I = [a,b] \) is an interval such that \( u((-\infty,a)) > 0 \), and if there exists an interval \( J = [b,c] \) such that \( \sigma(J) > 0 \), then for all \( x \in I \),

\[
M_+^+(\sigma \chi_{I \cup J})(x) \geq \sigma(J)/(c-a)^{1-\alpha}.
\]

Hence, by the \( (S_{p,q}^+) \) condition applied to \( I \cup J \),

\[
u(I) \leq C \sigma(I)^{1/p} (c-a)^{1-\alpha}/\sigma(J) < \infty.
\]

If no such \( J \) exists, then \( \sigma \equiv 0 \) on \([b,\infty)\), so \( v \equiv \infty \) on the same set. But if \( f \in L^p(v) \), then \( f \equiv 0 \) on \([b,\infty)\).

Below, we will apply Lemma 2.2 to closed intervals contained in an open set \( O_k \).

In \( O_k \), \( M_+^+ f > 0 \), so \( f \) cannot be identically zero to the right of these intervals. Further, by the definition of \( O_k \), \( u \) is not identically zero to the left of these intervals. Finally, since we will also be assuming that \( f \) has compact support, the intervals will be contained in some compact set. Hence Lemma 2.2 is applicable.

The last lemma is an extension of a result of Muckenhoupt [4] for Lebesgue measure. The proof of the extension is identical to his proof and so is omitted.

Lemma 2.3. Let \( \mu \) be a Borel measure, \( f \) a function, and \( \{I_\beta\} \) a collection of intervals, all contained in some interval \( I \), with the property that

\[
\int_{I_\beta} f \, d\mu \geq N \mu(I_\beta).
\]

If \( J = \bigcup_\beta I_\beta \) then

\[
\int_J f \, d\mu \geq (N/2) \mu(J).
\]

If \((u,v)\) satisfy the \( (S_{p,q}^+) \) condition, then \( \sigma \, dx = v^{1-p} \, dx \) is a Borel measure.

3. Proof of Theorem 1.1

To show that the \( (S_{p,q}^+) \) condition is necessary for inequality (1) of Theorem 1.1 to hold, first suppose that there is some interval \( I = [a,b] \) such that \( u((-\infty,a)) > 0 \) but \( \sigma(I) = \infty \). Equivalently, the function \( v^{-1} \chi_I \) is not in \( L^p(v) \), so there exists a function \( f \) in \( L^p(v) \) such that

\[
\infty = \int_I f v^{-1} \chi_I \, dx = \int_I f \, dx.
\]

Then for all \( x \in J \), \( M_+^+ f(x) = \infty \), which contradicts inequality (1). The rest of the \( (S_{p,q}^+) \) condition follows if we substitute \( f = \sigma \chi_I \) into the norm inequality.

To prove that the \( (S_{p,q}^+) \) condition is sufficient, we follow the outline of the proof of Martín-Reyes and de la Torre [3], which in turn is based on a proof by Sawyer [6]. Let \( f \) be in \( L^p(v) \); we will first consider the special case where \( f \) is a non-negative,
bounded, upper semicontinuous function of compact support. By Lemma 2.1, $M^+_\alpha f$ is continuous. Further, since the set $\{x : M^+_\alpha f(x) = \lambda\}$ has positive measure for at most a countable number of $\lambda$, by multiplying $f$ by a suitable constant we may assume without loss of generality that the sets $\{x : M^+_\alpha f(x) = 2^k\}$ have measure zero for all integers $k$.

Let $a = \sup\{x : u((-\infty, x)) = 0\}$. For each integer $k$ define the set $O_k = \{x : 2^k < M^+_\alpha f(x) < 2^{k+1}\} \cap (a, \infty)$. Since $M^+_\alpha f$ is continuous, it follows that each $O_k$ is open, and the set $\mathbb{R} \setminus \bigcup_k O_k$ has measure zero. For each $x \in O_k$, there exists an open interval $J_{x,k} = (x, t_x)$ such that

\[
2^k < \frac{1}{|J_{x,k}|^{1-\alpha}} \int_{J_{x,k}} f \, dy < 2^{k+1}.
\]

(2)

We claim that there exists a point $s_x \in J_{x,k}$ such that if $y \in I_{x,k} = [x, s_x]$ then

\[
\frac{1}{|J_{x,k}|^{1-\alpha}} \int_{J_{x,k}} \sigma \, dy \leq 2M^+\alpha(\sigma \chi_{J_{x,k}})(y).
\]

(3)

If $M^+_\alpha(\sigma \chi_{J_{x,k}})(x) = 0$ then this is immediate. If it is positive, then since $M^+_\alpha(\sigma \chi_{J_{x,k}})$ is lower semicontinuous, we can find $s_x$ such that $M^+_\alpha(\sigma \chi_{J_{x,k}})(y)$ is also positive for $y \in I_{x,k}$. By the continuity of the integral, the desired inequality holds if we take $s_x$ sufficiently close to $x$. Finally, since $O_k$ is open we may take $s_x$ so that $I_{x,k} \subset O_k$.

The union of the $I_{x,k}$’s is $O_k$. Therefore, by Lemma 2.2 and the remarks following it, there exists a finite, disjoint collection of intervals $\{I_{j,k}\}_{j=1}^{n_k}$ such that

\[
u(O_k) \leq 3 \sum_{j=1}^{n_k} \nu(I_{j,k}).
\]

(4)

Since the sets $O_k$ are disjoint, the intervals $I_{j,k}$ are pairwise disjoint for all $j$ and $k$.

Using inequalities (2) and (4), we can now make the following estimate:

\[
\int_{\mathbb{R}} (M^+_\alpha f)^q u \, dx = \sum_k \int_{O_k} (M^+_\alpha f)^q u \, dx \\
\leq \sum_k \nu(O_k) 2^{q(k+1)} \\
\leq C \sum_{j,k} \nu(I_{j,k}) 2^{qk} \\
\leq C \sum_{j,k} \nu(I_{j,k}) \left( \frac{1}{|J_{j,k}|^{1-\alpha}} \int_{J_{j,k}} f \, dx \right)^q \\
= C \sum_{j,k} \nu(I_{j,k}) \left( \frac{1}{|J_{j,k}|^{1-\alpha}} \int_{J_{j,k}} \sigma \, dx \right)^q \left( \frac{\int_{J_{j,k}} (f/\sigma) \cdot \sigma \, dx}{\int_{J_{j,k}} \sigma \, dx} \right)^q \left( \int_{J_{j,k}} \sigma \, dx \right)^q
\]

Define the measure $\omega$ on $X = \mathbb{N} \times \mathbb{Z}$ by

\[
\omega(j,k) = u(I_{j,k}) \left( \frac{1}{|J_{j,k}|^{1-\alpha}} \int_{J_{j,k}} \sigma \, dx \right)^q
\]
if \( j \leq n_k \) and \( \omega(j, k) = 0 \) if \( j > n_k \). Also define the operator \( T \) by

\[
T h(j, k) = \frac{\int_{J_{j,k}} |h| \sigma \, dx}{\int_{J_{j,k}} \sigma \, dx}.
\]

Then, following the argument of Sawyer, to get the desired norm inequality it will suffice to show that \( T \) is a bounded operator from \( L^p(\sigma) \) into \( L^q(X, \omega) \). Since \( T \) is bounded on \( L^\infty \), by Marcinkiewicz interpolation it will suffice to show that \( T \) is weak-type \((1, q/p)\): that is, for each \( \lambda > 0 \)

\[
\sum_{(j,k) \in E_\lambda} u(I_{j,k}) \left( \frac{1}{|J_{j,k}|^{1-\alpha}} \int_{J_{j,k}} \sigma \, dx \right)^q \leq C \left( \frac{1}{\lambda} \int_{\mathbb{R}} |h| \sigma \, dx \right)^{q/p},
\]

where \( E_\lambda = \{(j,k) \in X : T h(j, k) > \lambda \} \). If \( (j,k) \in E_\lambda \), then

\[
\int_{J_{j,k}} |h| \sigma \, dx > \lambda \int_{J_{j,k}} \sigma \, dx.
\]

Let \( G_\lambda \) be the union of all such \( J_{j,k} \)'s. Then, since the \( J_{j,k} \)'s are open, \( G_\lambda \) is the union of a countable number of disjoint open intervals \( J_i \). By Lemma 2.3, for each \( i \)

\[
\int_{J_i} |h| \sigma \, dx > \frac{\lambda}{2} \int_{J_i} \sigma \, dx. \tag{5}
\]

Since \( I_{j,k} \subset J_{j,k} \), each \( I_{j,k} \) is contained in exactly one interval \( J_i \). (Here we ignore the left endpoints of the \( I_{j,k} \)'s since they form a set of measure zero.) Therefore, by inequalities (3) and (5) and the \( (S^+_{p,q,\alpha}) \) condition, since \( q/p \geq 1 \), and since the \( I_{j,k} \)'s and the \( J_i \)'s are disjoint,

\[
\sum_{(j,k) \in E_\lambda} u(I_{j,k}) \left( \frac{1}{|J_{j,k}|^{1-\alpha}} \int_{J_{j,k}} \sigma \, dx \right)^q \leq 2 \sum_{(j,k) \in E_\lambda} \int_{I_{j,k}} M^+_\alpha(\sigma \chi_{J_{j,k}})^q u \, dx
\]

\[
\leq 2 \sum_{(j,k) \in E_\lambda} \sum_{I_{j,k} \subset J_i} \int_{I_{j,k}} M^+_\alpha(\sigma \chi_{J_i})^q u \, dx
\]

\[
\leq 2 \sum_i \int_{J_i} M^+_\alpha(\sigma \chi_{J_i})^q u \, dx
\]

\[
\leq C \sum_i \left( \int_{J_i} \sigma \, dx \right)^{q/p}
\]

\[
\leq C \sum_i \left( \frac{1}{\lambda} \int_{J_i} |h| \sigma \, dx \right)^{q/p}
\]

\[
\leq C \left( \frac{1}{\lambda} \int_{\mathbb{R}} |h| \sigma \, dx \right)^{q/p}.
\]

This completes the special case. To complete the proof, take any \( f \in L^p(v) \); then by the Vitali-Carathéodory theorem, there exists an increasing sequence \( \{f_n\} \) of non-negative, bounded, upper semicontinuous functions of compact support which converge to \( |f| \). (See, for example, Rudin [5, p. 57].) By the monotone convergence theorem, \( M^+_\alpha f_n \) increases pointwise to \( M^+_\alpha f \). Therefore, again by the monotone
convergence theorem,

\[
\left( \int_{\mathbb{R}} (M_\alpha^+ f)^q u \, dx \right)^{1/q} = \lim_{n \to \infty} \left( \int_{\mathbb{R}} (M_\alpha^+ f_n)^q u \, dx \right)^{1/q} \\
\leq \lim_{n \to \infty} C \left( \int_{\mathbb{R}} f_n^p v \, dx \right)^{1/p} \\
= C \left( \int_{\mathbb{R}} |f|^p v \, dx \right)^{1/p}.
\]

ACKNOWLEDGMENT

I want to thank F. Martín-Reyes and A. de la Torre for their hospitality, and my wife Gabrielle for her patient generosity; without either this paper would not have been possible.

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