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A NEW PROOF OF THE TWO WEIGHT NORM INEQUALITY FOR THE ONE-SIDED FRACTIONAL MAXIMAL OPERATOR

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ABSTRACT. We give a new proof of the two weight norm inequality for the one-sided, fractional maximal operator, M_α^+, simplifying the original proof of Martín-Reyes and de la Torre.

1. Introduction

In [1], Andersen and Sawyer introduced the one-sided fractional maximal operators

$$M_\alpha^+ f(x) = \sup_{t>0} \frac{1}{t^{1-\alpha}} \int_x^{x+t} |f|\,dy \quad \text{and} \quad M_\alpha^- f(x) = \sup_{t>0} \frac{1}{t^{1-\alpha}} \int_{x-t}^{x} |f|\,dy,$$

$0 < \alpha < 1$, in order to study the weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators. Using complex interpolation, they proved one-weight norm inequalities for M_α^+ and M_α^-. In [3], Martín-Reyes and de la Torre answered a question of Andersen and Sawyer by giving a geometric proof of a two-weight norm inequality for M_α^+. (The same result holds, *mutatis mutandis*, for M_α^-.)

Theorem 1.1. *For non-negative weights u and v, and for $1 < p \le q$, the following are equivalent:*

1. *There exists a constant C such that, for every function f in $L^p(v)$,*

$$\left(\int_{\mathbb{R}} (M_\alpha^+ f)^q u\,dx \right)^{1/q} \le C \left(\int_{\mathbb{R}} |f|^p v\,dx \right)^{1/p};$$

2. *The pair (u, v) satisfy the $(S^+_{p\,q\,\alpha})$ condition: there exists a constant C such that for every interval $I = [a, b]$ for which $u((-\infty, a)) > 0$,*

$$\left(\int_I M_\alpha^+ (\sigma \chi_I)^q u\,dx \right)^{1/q} \le C \left(\int_I \sigma\,dx \right)^{1/p} < \infty,$$

where $\sigma = v^{1-p'}$.

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Their proof involved proving a weighted norm inequality for a dyadic variant of the fractional maximal operator, $M^+_{\alpha\,D}$, using a dyadic version of the Sawyer condition, $(S^+_{p\,q\,\alpha\,D})$, and then showing that these were equivalent to M^+_α and $(S^+_{p\,q\,\alpha})$.

The purpose of this paper is to give a new proof of Theorem 1.1, one which eliminates the dyadic maximal operator. We do this by adapting the proof in the dyadic case using special covering properties of $\mathbb{R}$ and the continuity properties of the maximal operator. We believe that these techniques will be useful in proving other norm inequalities for maximal operators on $\mathbb{R}$.

The paper is organized as follows: Section 2 contains three lemmas and some remarks on their applicability, and Section 3 contains the actual proof. Throughout, all functions are assumed to be measurable, C denotes a positive constant whose value may be different at each appearance, $p' = p/(p-1)$ is the conjugate exponent of p, and $0 < \alpha < 1$. Given a Borel set E and a function w, let $|E|$ denote the Lebesgue measure of E and $w(E) = \int_E w\,dx$.

2. Preliminary results

Lemma 2.1. *Let f be a non-negative, bounded, upper semicontinuous function of compact support. Then $M^+_\alpha f$ is continuous.*

Proof. Since $M^+_\alpha f$ is always lower semicontinuous, it will suffice to show that it is upper semicontinuous at each point $x \in \mathbb{R}$. We will show this by contradiction: suppose that there exist an $\epsilon > 0$, a point x_0 and a sequence of points $\{x_n\}$ converging to x_0 such that $M^+_\alpha f(x_n) > M^+_\alpha f(x_0) + \epsilon$. For each n there exists an interval J_n whose left endpoint is x_n such that

$$\frac{1}{|J_n|^{1-\alpha}} \int_{J_n} |f|\,dx > M^+_\alpha f(x_0) + \epsilon. \tag{1}$$

Let the support of f be contained in the (finite) interval I, and let K be a finite, open interval containing I and the x_n's, $n \geq 0$. Then we may assume that each J_n is contained in K. Therefore, after passing to a subsequence we may assume that the J_n's converge to a possibly degenerate interval J whose left endpoint is x_0. If $|J| > 0$ then inequality (1) implies that

$$M^+_\alpha f(x_0) \geq \frac{1}{|J|^{1-\alpha}} \int_J |f|\,dx \geq M^+_\alpha f(x_0) + \epsilon,$$

a contradiction. If $J = \{x_0\}$ then for n sufficiently large, by the upper semicontinuity of f, $f(x) - f(x_0) < \epsilon$ for all $x \in J_n$. Therefore,

$$\frac{1}{|J_n|^{1-\alpha}} \int_{J_n} f(x)\,dx < |J_n|^\alpha (f(x_0) + \epsilon).$$

If we combine this with inequality (1) and take the limit as n tends to infinity, we get (since f is bounded) that $0 \geq M^+_\alpha f(x_0) + \epsilon$, which is again a contradiction. Hence $M^+_\alpha f$ is upper semicontinuous at each point, and we are done. □

It is worth noting that this lemma is not true if $\alpha = 0$. A simple counter-example is given by the characteristic function of $[0, 1]$. (I am grateful to Juha Kinnunen for pointing this out to me.)

The next lemma is due to Jesus Aldaz; the proof is in Bliedtner and Loeb [2].

Lemma 2.2. *If μ is a finite Borel measure on $\mathbb{R}$, and if $\mathcal{I}$ is an arbitrary collection of non-degenerate intervals, then for each $\delta > 0$ there exists a finite subcollection, $\mathcal{I}_\delta$, of disjoint intervals in $\mathcal{I}$ such that*

$$\mu\Big(\bigcup_{I\in\mathcal{I}} I\Big) \le (2+\delta) \sum_{I_k\in\mathcal{I}_\delta} \mu(I_k).$$

Below we will want to apply Lemma 2.2 with the measure $u\,dx$, where (u, v) satisfies the $(S^+_{p\,q\,\alpha})$ condition. To do this, we need u to be locally integrable and the intervals to be contained in some compact set. However, if $I = [a, b]$ is an interval such that $u((-\infty, a)) > 0$, and if there exists an interval $J = [b, c]$ such that $\sigma(J) > 0$, then for all $x \in I$,

$$M^+_\alpha(\sigma\chi_{I\cup J})(x) \ge \sigma(J)/(c-a)^{1-\alpha}.$$

Hence, by the $(S^+_{p\,q\,\alpha})$ condition applied to $I \cup J$,

$$u(I) \le C\sigma(I)^{q/p}(c-a)^{1-\alpha}/\sigma(J) < \infty.$$

If no such J exists, then $\sigma \equiv 0$ on $[b, \infty)$, so $v \equiv \infty$ on the same set. But if $f \in L^p(v)$, then $f \equiv 0$ on $[b, \infty)$.

Below, we will apply Lemma 2.2 to closed intervals contained in an open set O_k. In O_k, $M^+_\alpha f > 0$, so f cannot be identically zero to the right of these intervals. Further, by the definition of O_k, u is not identically zero to the left of these intervals. Finally, since we will also be assuming that f has compact support, the intervals will be contained in some compact set. Hence Lemma 2.2 is applicable.

The last lemma is an extension of a result of Muckenhoupt [4] for Lebesgue measure. The proof of the extension is identical to his proof and so is omitted.

Lemma 2.3. *Let μ be a Borel measure, f a function, and $\{I_\beta\}$ a collection of intervals, all contained in some interval I, with the property that*

$$\int_{I_\beta} f\,d\mu \ge N\mu(I_\beta).$$

If $J = \bigcup_\beta I_\beta$ then

$$\int_J f\,d\mu \ge (N/2)\mu(J).$$

If (u, v) satisfy the $(S^+_{p\,q\,\alpha})$ condition, then $\sigma\,dx = v^{1-p'}\,dx$ is a Borel measure.

3. Proof of Theorem 1.1

To show that the $(S^+_{p\,q\,\alpha})$ condition is necessary for inequality (1) of Theorem 1.1 to hold, first suppose that there is some interval $I = [a, b]$ such that $u((-\infty, a)) > 0$ but $\sigma(I) = \infty$. Equivalently, the function $v^{-1}\chi_I$ is not in $L^{p'}(v)$, so there exists a function f in $L^p(v)$ such that

$$\infty = \int_I f v^{-1} v\,dx = \int_I f\,dx.$$

Then for all $x \in J$, $M^+_\alpha f(x) = \infty$, which contradicts inequality (1). The rest of the $(S^+_{p\,q\,\alpha})$ condition follows if we substitute $f = \sigma\chi_I$ into the norm inequality.

To prove that the $(S^+_{p\,q\,\alpha})$ condition is sufficient, we follow the outline of the proof of Martín-Reyes and de la Torre [3], which in turn is based on a proof by Sawyer [6]. Let f be in $L^p(v)$; we will first consider the special case where f is a non-negative, bounded, upper semicontinuous function of compact support. By Lemma 2.1, $M_\alpha^+ f$ is continuous. Further, since the set $\{x : M_\alpha^+ f(x) = \lambda\}$ has positive measure for at most a countable number of λ, by multiplying f by a suitable constant we may assume without loss of generality that the sets $\{x : M_\alpha^+ f(x) = 2^k\}$ have measure zero for all integers k.

Let $a = \sup\{x : u((-\infty, x)) = 0\}$. For each integer k define the set $O_k = \{x : 2^k < M_\alpha^+ f(x) < 2^{k+1}\} \cap (a, \infty)$. Since $M_\alpha^+ f$ is continuous, it follows that each O_k is open, and the set $\mathbb{R} \setminus \bigcup_k O_k$ has measure zero. For each $x \in O_k$, there exists an open interval $J_{x\,k} = (x, t_x)$ such that

$$2^k < \frac{1}{|J_{x\,k}|^{1-\alpha}} \int_{J_{x\,k}} f\,dy < 2^{k+1}. \tag{2}$$

We claim that there exists a point $s_x \in J_{x\,k}$ such that if $y \in I_{x\,k} = [x, s_x]$ then

$$\frac{1}{|J_{x\,k}|^{1-\alpha}} \int_{J_{x\,k}} \sigma\,dy \le 2M_\alpha^+(\sigma\chi_{J_{x\,k}})(y). \tag{3}$$

If $M_\alpha^+(\sigma\chi_{J_{x\,k}})(x) = 0$ then this is immediate. If it is positive, then since $M_\alpha^+(\sigma\chi_{J_{x\,k}})$ is lower semicontinuous, we can find s_x such that $M_\alpha^+(\sigma\chi_{J_{x\,k}})(y)$ is also positive for $y \in I_{x\,k}$. By the continuity of the integral, the desired inequality holds if we take s_x sufficiently close to x. Finally, since O_k is open we may take s_x so that $I_{x\,k} \subset O_k$.

The union of the $I_{x\,k}$'s is O_k. Therefore, by Lemma 2.2 and the remarks following it, there exists a finite, disjoint collection of intervals $\{I_{j\,k}\}_{j=1}^{n_k}$ such that

$$u(O_k) \le 3 \sum_{j=1}^{n_k} u(I_{j\,k}). \tag{4}$$

Since the sets O_k are disjoint, the intervals $I_{j\,k}$ are pairwise disjoint for all j and k.

Using inequalities (2) and (4), we can now make the following estimate:

$$\begin{aligned}
\int_{\mathbb{R}} (M_\alpha^+ f)^q u\,dx &= \sum_k \int_{O_k} (M_\alpha^+ f)^q u\,dx \\
&\le \sum_k u(O_k) 2^{q(k+1)} \\
&\le C \sum_{j,k} u(I_{j\,k}) 2^{qk} \\
&\le C \sum_{j,k} u(I_{j\,k}) \left(\frac{1}{|J_{j\,k}|^{1-\alpha}} \int_{J_{j\,k}} f\,dx \right)^q \\
&= C \sum_{j,k} u(I_{j\,k}) \left(\frac{1}{|J_{j\,k}|^{1-\alpha}} \int_{J_{j\,k}} \sigma\,dx \right)^q \left(\frac{\int_{J_{j\,k}} (f/\sigma)\cdot\sigma\,dx}{\int_{J_{j\,k}} \sigma\,dx} \right)^q
\end{aligned}$$

Define the measure ω on $X = \mathbb{N} \times \mathbb{Z}$ by

$$\omega(j,k) = u(I_{j\,k}) \left(\frac{1}{|J_{j\,k}|^{1-\alpha}} \int_{J_{j\,k}} \sigma\,dx \right)^q$$

if $j \le n_k$, and $\omega(j,k) = 0$ if $j > n_k$. Also define the operator T by

$$Th(j,k) = \frac{\int_{J_{j\,k}} |h|\sigma\,dx}{\int_{J_{j\,k}} \sigma\,dx}.$$

Then, following the argument of Sawyer, to get the desired norm inequality it will suffice to show that T is a bounded operator from $L^p(\sigma)$ into $L^q(X,\omega)$. Since T is bounded on L^∞, by Marcinkiewicz interpolation it will suffice to show that T is weak-type $(1, q/p)$: that is, for each $\lambda > 0$

$$\sum_{(j,k)\in E_\lambda} u(I_{j\,k}) \left(\frac{1}{|J_{j\,k}|^{1-\alpha}} \int_{J_{j\,k}} \sigma\,dx \right)^q \le C \left(\frac{1}{\lambda} \int_{\mathbb{R}} |h|\sigma\,dx \right)^{q/p},$$

where $E_\lambda = \{(j,k) \in X : Th(j,k) > \lambda\}$. If $(j,k) \in E_\lambda$, then

$$\int_{J_{j\,k}} |h|\sigma\,dx > \lambda \int_{J_{j\,k}} \sigma\,dx.$$

Let G_λ be the union of all such $J_{j\,k}$'s. Then, since the $J_{j\,k}$'s are open, G_λ is the union of a countable number of disjoint open intervals J_i. By Lemma 2.3, for each i

$$\int_{J_i} |h|\sigma\,dx > \frac{\lambda}{2} \int_{J_i} \sigma\,dx. \tag{5}$$

Since $I_{j\,k} \subset J_{j\,k}$, each $I_{j\,k}$ is contained in exactly one interval J_i. (Here we ignore the left endpoints of the $I_{j\,k}$'s since they form a set of measure zero.) Therefore, by inequalities (3) and (5) and the $(S^+_{p\,q\,\alpha})$ condition, since $q/p \ge 1$, and since the $I_{j\,k}$'s and the J_i's are disjoint,

$$\begin{aligned}
\sum_{(j,k)\in E_\lambda} u(I_{j\,k}) \left(\frac{1}{|J_{j\,k}|^{1-\alpha}} \int_{J_{j\,k}} \sigma\,dx \right)^q &\le 2 \sum_{(j,k)\in E_\lambda} \int_{I_{j\,k}} M^+_\alpha(\sigma\chi_{J_{j\,k}})^q u\,dx \\
&\le 2 \sum_i \sum_{I_{j\,k}\subset J_i} \int_{I_{j\,k}} M^+_\alpha(\sigma\chi_{J_i})^q u\,dx \\
&\le 2 \sum_i \int_{J_i} M^+_\alpha(\sigma\chi_{J_i})^q u\,dx \\
&\le C \sum_i \left(\int_{J_i} \sigma\,dx \right)^{q/p} \\
&\le C \sum_i \left(\frac{1}{\lambda} \int_{J_i} |h|\sigma\,dx \right)^{q/p} \\
&\le C \left(\frac{1}{\lambda} \int_{\mathbb{R}} |h|\sigma\,dx \right)^{q/p}.
\end{aligned}$$

This completes the special case. To complete the proof, take any $f \in L^p(v)$; then by the Vitali-Carathéodory theorem, there exists an increasing sequence $\{f_n\}$ of non-negative, bounded, upper semicontinuous functions of compact support which converge to $|f|$. (See, for example, Rudin [5, p. 57].) By the monotone convergence theorem, $M^+_\alpha f_n$ increases pointwise to $M^+_\alpha f$. Therefore, again by the monotone convergence theorem,

$$\begin{aligned}\left(\int_{\mathbb{R}}(M_\alpha^+ f)^q u\,dx\right)^{1/q} &= \lim_{n\to\infty}\left(\int_{\mathbb{R}}(M_\alpha^+ f_n)^q u\,dx\right)^{1/q}\\ &\le \lim_{n\to\infty} C\left(\int_{\mathbb{R}} f_n^p v\,dx\right)^{1/p}\\ &= C\left(\int_{\mathbb{R}}|f|^p v\,dx\right)^{1/p}.\end{aligned}$$

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References

[1] K. F. Andersen and E. T. Sawyer, *Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators*, Trans. Amer. Math. Soc. 308 (1988), 547-557. MR **89h:**26006

[2] J. Bliedtner and P. Loeb, *A reduction technique for limit theorems in analysis and probability theory,* Ark. Mat. 30 (1992), 25-43. MR **93k:**28005

[3] F. J. Martín-Reyes and A. de la Torre, *Two weight norm inequalities for fractional one-sided maximal operators*, Proc. Amer. Math. Soc. 117 (1993), 483-489. MR **94b:**42010

[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function,* Trans. Amer. Math. Soc. 165 (1972), 207-226. MR **45:**4261

[5] W. Rudin, *Real and Complex Analysis,* McGraw-Hill, New York, 1974. MR **49:**8783

[6] E. T. Sawyer, *A characterization of a two-weight norm inequality for maximal operators*, Studia Math. 75 (1982), 1-11. MR **84i:**42032

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