Approximation of Fixed Points of a Strictly Pseudocontractive Mapping

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Abstract. A fixed point of the strictly pseudocontractive mapping is obtained as the limit of an iteratively constructed sequence with error estimation in general Banach spaces.

Let $X$ be a Banach space. A mapping $T$ is said to be strictly pseudocontractive if there exists a number $t > 1$ such that the inequality

$$
\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|
$$

holds for all $x, y \in D(T)$ and $r > 0$. We denote by $J$ the normalized duality mapping from $X$ to $2^X^*$ given by

$$
Jx = \{f^\ast \in X^* : \|f^\ast\|^2 = \|x\|^2 = \text{Re}\langle x, f^\ast \rangle\}.
$$

A mapping $T$ is said to be strongly accretive (see, e.g., [1, 2]) if there exists a positive number $k$ such that for each $x, y \in D(T)$ there is $j \in J(x - y)$ such that

$$
\langle Tx - Ty, j \rangle \geq k\|x - y\|^2.
$$

(1)

The inequality (1) implies the inequality

$$
\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\|.
$$

Without loss of generality, we can assume $k \in (0, 1)$.

Lemma ([6]). Let $X$ be a Banach space, $K$ a subset of $X$, and $T : K \rightarrow K$. Then $T$ is a strictly pseudocontractive mapping if and only if $I - T$ is a strongly accretive mapping, i.e., the inequality

$$
\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\|
$$

holds for any $x, y \in K$ and $r > 0$, where $k = (t - 1)/t$.

In this paper, we prove by the inequality (2) that the Mann iteration process converges strongly to the unique fixed point of a Lipschitzian and strictly pseudocontractive mapping. Our results extend corresponding results of [1, 2, 3, 4, 5] to the general Banach spaces. Furthermore, error estimates are also given.

The set of fixed points of the mapping $T$ is denoted by $F(T)$. The Lipschitzian constant of $T$ is denoted by $L (\geq 1)$.

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Theorem 1. Let $X$ be a Banach space, and let $K$ be a nonempty closed convex and bounded subset of $X$. Let $T : K \rightarrow K$ be a Lipschitzian strictly pseudocontractive mapping. If $F(T) \neq \emptyset$, then $\{x_n\} \subset K$ generated by $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + a_nTx_n$$

with $\{\alpha_n\} \subset (0, 1]$ satisfying

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0,$$

strongly converges to $q \in F(T)$ and $F(T)$ is a single set.

Proof. Let $q$ be a fixed point of $T$. Since $T$ is a strictly pseudocontractive mapping, then $I - T$ is strongly accretive, i.e., the inequality (2) holds for any $x, y \in K$ and $r > 0$.

From the definition of $\{x_n\}$, we have

\[
x_n = x_{n+1} + \alpha_n x_n - \alpha_n Tx_n \\
= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (2 - k)\alpha_nx_{n+1} \\
+ \alpha_n x_n + \alpha_n(Tx_{n+1} - Tx_n) \\
= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (2 - k)\alpha_n[(1 - \alpha_n)x_n + \alpha_nTx_n] \\
+ \alpha_n x_n + \alpha_n(Tx_{n+1} - Tx_n) \\
= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (1 - k)\alpha_n x_n + (2 - k)\alpha_n^2(x_n - Tx_n) \\
+ \alpha_n(Tx_{n+1} - Tx_n).
\]

By $Tq = q$, we have

\[
x_n - q = (1 + \alpha_n)(x_{n+1} - q) + \alpha_n(I - T - kI)(x_{n+1} - q) - (1 - k)\alpha_n(x_n - q) \\
+ (2 - k)\alpha_n^2(x_n - Tx_n) + \alpha_n(Tx_{n+1} - Tx_n).
\]

By using the inequality (2), we obtain

\[
\|x_n - q\| \geq (1 + \alpha_n)\|x_{n+1} - q\| - (1 - k)\alpha_n\|x_n - q\| - (2 - k)\alpha_n^2\|x_n - Tx_n\| \\
- \alpha_n\|Tx_{n+1} - Tx_n\|.
\]

Since

\[
\|Tx_{n+1} - Tx_n\| \leq L\|x_{n+1} - x_n\| \leq L(L + 1)\alpha_n\|x_n - q\|,
\]

then

\[
\|x_n - q\| \geq (1 + \alpha_n)\|x_{n+1} - q\| - (1 - k)\alpha_n\|x_n - q\| - (2 - k)\alpha_n^2\|x_n - Tx_n\| \\
- L(L + 1)\alpha_n^2\|x_n - q\|.
\]
So,
\[
\|x_{n+1} - q\| \leq [1 + (1 - k)\alpha_n](1 + \alpha_n)^{-1}\|x_n - q\| + (2 - k)\alpha_n^2(1 + \alpha_n)^{-1}\|x_n - Tx_n\| + L(L + 1)\alpha_n^2(1 + \alpha_n)^{-1}\|x_n - q\| \\
\leq [1 + (1 - k)\alpha_n](1 - \alpha_n + \alpha_n^2)\|x_n - q\| + (2 - k)\alpha_n\|x_n - Tx_n\| + L(L + 1)\alpha_n^2\|x_n - q\| \\
\leq (1 - \kappa_\alpha)|x_n - q| + M\alpha_n^2 
\]  
for some constant \(M > 0\), since \(K\) is bounded. It follows from the Lemma of Dunn [7, p. 41] that the sequence \(\{x_n\}\) strongly converges to the unique fixed point \(q\). This completes the proof.

**Remark.** The nonlipschitzian version of Theorem 1 was proved in [1, 3] under the assumption that \(X\) is a uniformly smooth Banach space by using the inequality

\[
\|x + y\|^2 \leq \|x\|^2 + 2 \text{Re}(y, Jx) + \max\{\|x\|, 1\}\|y\|\beta(\|y\|).
\]

**Theorem 2.** Let \(K\) and \(T\) be as in Theorem 1. If \(\alpha_n = \frac{k}{2(3+3L+L^2)}\), where \(k = \frac{(t - 1)/t\cdot \{q\}}{F(T)}\), then the sequence \(\{x_n\}\) generated by (3) converges strongly to the unique fixed point of \(T\), and we have the estimate

\[
\|x_{n+1} - q\| \leq \rho^n\|x_1 - q\|,
\]

where \(\rho = 1 - \frac{k^2}{4(3+3L+L^2)}\).

**Proof.** Since \(0 < \alpha_n < 1\) and \(Tq = q\),

\[
[1 + (1 - k)\alpha_n](1 - \alpha_n + \alpha_n^2) = 1 - k\alpha_n + \alpha_n^2 - (1 - k)\alpha_n^2(1 - \alpha_n) \leq 1 - k\alpha_n + \alpha_n^2.
\]

\[
\|x_n - Tx_n\| \leq (1 + L)\|x_n - q\|.
\]

By (4), we obtain

\[
\|x_{n+1} - q\| \leq (1 - \kappa_\alpha)|x_n - q| + [1 + (2 - k)(1 + L) + L(L + 1)]\alpha_n^2\|x_n - q\| \\
< [1 - \kappa_\alpha + (3 + 3L + L^2)\alpha_n^2]\|x_n - q\| \\
= \left[1 - \frac{k^2}{2(3+3L+L^2)} + \frac{k^2}{4(3+3L+L^2)}\right]\|x_n - q\| \\
= \rho\|x_n - q\|.
\]

Hence \(\|x_{n+1} - q\| < \rho^n\|x_1 - q\|\). The proof is complete.

**References**


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