PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 5, May 1997, Pages 1363–1366
S 0002-9939(97)03858-6

# APPROXIMATION OF FIXED POINTS OF A STRICTLY PSEUDOCONTRACTIVE MAPPING

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. A fixed point of the strictly pseudocontractive mapping is obtained as the limit of an iteratively constructed sequence with error estimation in general Banach spaces.

Let $X$ be a Banach space. A mapping $T$ is said to be strictly pseudocontractive if there exists a number $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

holds for all $x, y \in D(T)$ and $r > 0$. We denote by $J$ the normalized duality mapping from $X$ to $2^{X^*}$ given by

$$Jx = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = \mathrm{Re}\langle x, f^*\rangle\}.$$

A mapping $T$ is said to be strongly accretive (see, e.g., [1, 2]) if there exists a positive number $k$ such that for each $x, y \in D(T)$ there is $j \in J(x - y)$ such that

$$\langle Tx - Ty, j\rangle \geq k\|x - y\|^2. \tag{1}$$

The inequality (1) implies the inequality

$$\|x - y\| \leq \|x - y + r[(T - kI)x - (T - kI)y]\|.$$

Without loss of generality, we can assume $k \in (0, 1)$.

**Lemma** ([6])**.** *Let $X$ be a Banach space, $K$ a subset of $X$, and $T : K \to K$. Then $T$ is a strictly pseudocontractive mapping if and only if $I - T$ is a strongly accretive mapping, i.e., the inequality*

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\| \tag{2}$$

*holds for any $x, y \in K$ and $r > 0$, where $k = (t - 1)/t$.*

In this paper, we prove by the inequality (2) that the Mann iteration process converges strongly to the unique fixed point of a Lipschitzian and strictly pseudocontractive mapping. Our results extend corresponding results of [1, 2, 3, 4, 5] to the general Banach spaces. Furthermore, error estimates are also given.

The set of fixed points of the mapping $T$ is denoted by $F(T)$. The Lipschitzian constant of $T$ is denoted by $L$ $(\geq 1)$.

Received by the editors July 11, 1995 and, in revised form, November 3, 1995.
1991 *Mathematics Subject Classification.* Primary 47H17.
*Key words and phrases.* Strictly pseudocontractive mapping, iterative process.

**Theorem 1.** *Let $X$ be a Banach space, and let $K$ be a nonempty closed convex and bounded subset of $X$. Let $T : K \to K$ be a Lipschitzian strictly pseudocontractive mapping. If $F(T) \neq \varnothing$, then $\{x_n\} \subset K$ generated by $x_1 \in K$,*

$$x_{n+1} = (1 - \alpha_n)x_n + a_n T x_n \tag{3}$$

*with $\{\alpha_n\} \subset (0, 1]$ satisfying*

$$\sum_{n=1}^{\infty} a_n = \infty, \qquad \alpha_n \to 0,$$

*strongly converges to $q \in F(T)$ and $F(T)$ is a single set.*

*Proof.* Let $q$ be a fixed point of $T$. Since $T$ is a strictly pseudocontractive mapping, then $I - T$ is strongly accretive, i.e., the inequality (2) holds for any $x, y \in K$ and $r > 0$.

From the definition of $\{x_n\}$, we have

$$\begin{aligned}
x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T x_n \\
&= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (2 - k)\alpha_n x_{n+1} \\
&\qquad + \alpha_n x_n + \alpha_n(Tx_{n+1} - Tx_n) \\
&= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (2 - k)\alpha_n[(1 - \alpha_n)x_n + \alpha_n T x_n] \\
&\qquad + \alpha_n x_n + \alpha_n(Tx_{n+1} - Tx_n) \\
&= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (1 - k)\alpha_n x_n + (2 - k)\alpha_n^2(x_n - Tx_n) \\
&\qquad + \alpha_n(Tx_{n+1} - Tx_n).
\end{aligned}$$

By $Tq = q$, we have

$$\begin{aligned}
x_n - q = (1 + \alpha_n)(x_{n+1} - q) + \alpha_n(I - T - kI)(x_{n+1} - q) - (1 - k)\alpha_n(x_n - q) \\
+ (2 - k)\alpha_n^2(x_n - Tx_n) + \alpha_n(Tx_{n+1} - Tx_n).
\end{aligned}$$

By using the inequality (2), we obtain

$$\begin{aligned}
\|x_n - q\| \geq (1 + \alpha_n)\|x_{n+1} - q\| - (1 - k)\alpha_n\|x_n - q\| - (2 - k)\alpha_n^2\|x_n - Tx_n\| \\
- \alpha_n\|Tx_{n+1} - Tx_n\|.
\end{aligned}$$

Since

$$\|Tx_{n+1} - Tx_n\| \leq L\|x_{n+1} - x_n\| \leq L(L + 1)\alpha_n\|x_n - q\|,$$

then

$$\begin{aligned}
\|x_n - q\| \geq (1 + \alpha_n)\|x_{n+1} - q\| - (1 - k)\alpha_n\|x_n - q\| - (2 - k)\alpha_n^2\|x_n - Tx_n\| \\
- L(L + 1)\alpha_n^2\|x_n - q\|.
\end{aligned}$$

So,

$$
\begin{aligned}
\|x_{n+1}-q\| &\le [1+(1-k)\alpha_n](1+\alpha_n)^{-1}\|x_n-q\| \\
&\qquad + (2-k)\alpha_n^2(1+\alpha_n)^{-1}\|x_n-Tx_n\| \\
&\qquad + L(L+1)\alpha_n^2(1+\alpha_n)^{-1}\|x_n-q\| \\
&\le [1+(1-k)\alpha_n](1-\alpha_n+\alpha_n^2)\|x_n-q\| + (2-k)\alpha_n^2\|x_n-Tx_n\| \\
&\qquad + L(L+1)\alpha_n^2\|x_n-q\| \\
&\le (1-k\alpha_n)\|x_n-q\| + M\alpha_n^2
\end{aligned}
\tag{4}
$$

for some constant $M>0$, since $K$ is bounded. It follows from the Lemma of Dunn [7, p. 41] that the sequence $\{x_n\}$ strongly converges to the unique fixed point $q$. This completes the proof.

*Remark.* The nonlipschitzian version of Theorem 1 was proved in [1, 3] under the assumption that $X$ is a uniformly smooth Banach space by using the inequality

$$\|x+y\|^2 \le \|x\|^2 + 2\,\mathrm{Re}\langle y, Jx\rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|).$$

**Theorem 2.** *Let $K$ and $T$ be as in Theorem 1. If $\alpha_n = \frac{k}{2(3+3L+L^2)}$, where $k=(t-1)/t$, $\{q\}=F(T)$, then the sequence $\{x_n\}$ generated by* (3) *converges strongly to the unique fixed point of $T$, and we have the estimate*

$$\|x_{n+1}-q\| < \rho^n\|x_1-q\|,$$

*where $\rho = 1-\frac{k^2}{4(3+3L+L^2)}$.*

*Proof.* Since $0<\alpha_n<1$ and $Tq=q$,

$$
\begin{aligned}
[1+(1-k)\alpha_n](1-\alpha_n+\alpha_n^2) &= 1-k\alpha_n+\alpha_n^2-(1-k)\alpha_n^2(1-\alpha_n) \\
&\le 1-k\alpha_n+\alpha_n^2,
\end{aligned}
$$

$$\|x_n-Tx_n\| \le (1+L)\|x_n-q\|.$$

By (4), we obtain

$$
\begin{aligned}
\|x_{n+1}-q\| &\le (1-k\alpha_n)\|x_n-q\| + [1+(2-k)(1+L)+L(L+1)]\alpha_n^2\|x_n-q\| \\
&< [1-k\alpha_n+(3+3L+L^2)\alpha_n^2]\|x_n-q\| \\
&= \left[1-\frac{k^2}{2(3+3L+L^2)}+\frac{k^2}{4(3+3L+L^2)}\right]\|x_n-q\| \\
&= \left[1-\frac{k^2}{4(3+3L+L^2)}\right]\|x_n-q\| \\
&= \rho\|x_n-q\|.
\end{aligned}
$$

Hence $\|x_{n+1}-q\| < \rho^n\|x_1-q\|$. The proof is complete.

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