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AN APPROXIMATION CONDITION AND EXTREMAL QUASICONFORMAL EXTENSIONS

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ABSTRACT. The possibility that the extremal dilatation of quasiconformal extensions from the circle is determined by quadrilaterals with vertices on the circle is related to an approximation question for holomorphic functions. This allows an alternative demonstration of a result of Anderson and Hinkkanen.

1. INTRODUCTION

Let D denote the unit disk in the complex plane, and let f be a sense-preserving quasisymmetric homeomorphism of the unit circle ∂D onto itself. In a recent paper [1] Anderson and Hinkkanen considered, following their notation, two constants, $K_0(f)$, and $K_1(f)$, associated with f, defined as follows:

$$K_0(f) = \sup_Q \frac{M(f(Q))}{M(Q)}, \tag{1.1}$$

and

$$K_1(f) = \inf\{K : f \text{ has a } K\text{-quasiconformal extension to } D \cup \partial D\}. \tag{1.2}$$

In (1.1) Q denotes a quadrilateral $D(z_1, z_2, z_3, z_4)$ with domain D and vertices $z_n \in \partial D$, and $M(Q)$ its conformal modulus. The supremum in (1.1) is over all choices of the four vertices z_1, z_2, z_3, z_4, on ∂D. From the basic definition of quasiconformality it is of course clear that

$$K_0(f) \le K_1(f). \tag{1.3}$$

It was shown in [1] that there exist f for which the inequality (1.3) is *strict*. Our objective is to give a proof of the result of Anderson and Hinkkanen by showing how it is related to an approximation question for holomorphic functions.

We recall the well known fact [3] that a self-mapping f of D with complex dilatation

$$\mu(z) = \frac{f_{\bar z}}{f_z} = t\frac{\overline{\phi(z)}}{|\phi(z)|}, \quad \text{for a.a. } z \in D, \tag{1.4}$$

where $0 < t < 1$, and ϕ is holomorphic in D and in class $L^1(D)$, is extremal for its induced boundary values; that is,

$$K_1(f) = \frac{1+t}{1-t}.$$

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Without loss of generality, we assume that $\|\phi\| = \iint_D |\phi(z)|\, dx\, dy = 1$.

Theorem 1. *A necessary condition that $K_0(f) = K_1(f)$ for boundary values induced by a mapping of type (1.4) is that ϕ can be approximated by squares of holomorphic functions sufficiently well so that*

$$\sup_{\Phi} \left| \iint_D \frac{\overline{\phi(z)}}{|\phi(z)|} \Phi'(z)^2\, dx\, dy \right| = 1. \tag{1.5}$$

The supremum in (1.5) is taken over all functions Φ holomorphic in D for which

$$\iint_D |\Phi'(z)|^2\, dx\, dy = 1. \tag{1.6}$$

.

2. Proof of the theorem

The proof, presented here for the sake of completeness, follows the simplest version of the length-area method on the model of [2, Section 3], but it is more elementary than the analysis in [2] in that nothing about quadratic differentials is needed.

We first transfer Q to a rectangle

$$R = \{\zeta = \xi + i\eta : 0 \le \xi \le a, 0 \le \eta \le b\}, \quad |R| = ab = 1,$$

by a conformal mapping $\zeta = \Phi_Q(z)$ in such a manner that the vertices (z_1, z_2, z_3, z_4) are mapped onto the four vertices of R. By Schwarz-Christoffel,

$$\Phi_Q'(z)^2 = \frac{A}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)},$$

where $A = A(z_1, z_2, z_3, z_4)$ is determined by the requirement that $|R| = ab = 1$. Actually, however, the exact form of $\Phi_Q(z)$ will not be relevant. Similarly, let $\Psi_Q(w)$ map $f(Q)$ conformally onto a rectangle R' of width a' and height b' so that the vertices of $f(Q)$ map onto the vertices of R'. Let

$$K_Q = \frac{1 + k_Q}{1 - k_Q} = \frac{M(f(Q))}{M(Q)} = \frac{a'/b'}{a/b}. \tag{2.1}$$

The induced map $g = \Psi \circ f \circ \Phi^{-1}$ maps R onto R' so that the vertices correspond, and has complex dilatation

$$\kappa(\zeta) = \frac{g_{\bar\zeta}}{g_\zeta} = \mu(z) \frac{\Phi_Q'(z)}{\overline{\Phi_Q'(z)}} \quad (\zeta = \Phi_Q(z)).$$

Since the images of horizontal cross-sections of R under g are at least of length a', we have

$$a'b \le \iint_R |g_\zeta + g_{\bar\zeta}|\, d\xi\, d\eta.$$

Inserting the factor $\sqrt{J}/\sqrt{J}$ in the integrand, with the Jacobian $J = |g_\zeta|^2 - |g_{\bar\zeta}|^2$, one obtains, by Schwarz's inequality,

$$(a'b)^2 \le a'b' \iint_R \frac{|1 + \kappa|^2}{1 - |\kappa|^2}\, d\xi\, d\eta.$$

By (2.1), we can rewrite this as

$$\frac{k_Q}{1-k_Q} \le \Re \iint_R \frac{\kappa}{1-|\kappa|^2}\, d\xi\, d\eta + \iint_R \frac{|\kappa|^2}{1-|\kappa|^2}\, d\xi\, d\eta.$$

Going back to the z-plane, this becomes

$$\begin{aligned} \frac{k_Q}{1-k_Q} &\le \Re \iint_D \frac{\mu \Phi_Q'^2}{1-|\mu|^2}\, dx\, dy + \iint_D \frac{|\mu|^2}{1-|\mu|^2} |\Phi_Q'|^2\, dx\, dy \\ &= \frac{t^2}{1-t^2} + \frac{t}{1-t^2} \Re \iint_D \frac{\overline{\phi}}{|\phi|} \Phi_Q'^2\, dx\, dy. \end{aligned} \tag{2.2}$$

Our hypothesis is that $\sup_Q k_Q = t$. Hence, (2.2) implies that

$$\sup_Q \left\{ \Re \iint_D \frac{\overline{\phi}}{|\phi|} \Phi_Q'^2\, dx\, dy \right\} \ge 1.$$

Therefore, (1.5) follows whenever Φ is a function holomorphic in D, satisfying (1.6).

3. An example

In order to exploit Theorem 1 to give an example for which

$$K_0(f) < K_1(f) \tag{3.1}$$

one evidently needs a function $\phi(z)$ whose square root is not single-valued in D. If we take

$$\phi(z) = \frac{3}{2\pi} z \tag{3.2}$$

and assume

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \pi \sum_{n=1}^{\infty} n|a_n|^2 = 1,$$

we obtain

$$\left| \iint_D \frac{\overline{\phi(z)}}{|\phi(z)|} \Phi'(z)^2\, dx\, dy \right| = \frac{8\pi}{3} |a_1 a_2|, \quad \pi(|a_1|^2 + 2|a_2|^2) \le 1.$$

Since

$$\max_\Phi \left| \iint_D \frac{\overline{\phi(z)}}{|\phi(z)|} \Phi'(z)^2\, dx\, dy \right| = \frac{2\sqrt{2}}{3} = 0.9428...$$

is attained, condition (1.5) is violated, and therefore (3.1) holds.

References

1. J. M. Anderson and A. Hinkkanen, *Quadrilaterals and extremal quasiconformal extensions*, Comment. Math. Helv. **70** (1995), 455–474. MR **96g:**30042
2. Edgar Reich and Kurt Strebel, *Extremal quasiconformal mappings with given boundary values*, Contributions to Analysis: A Collection of Papers Dedicated to Lipman Bers, Academic Press, New York, 1974, pp. 375–391. MR **50:**13511
3. Kurt Strebel, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises II*, Comment. Math. Helv. **39** (1964), 77–89. MR **31:**346

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