

EXTREME POINTS OF UNIT BALLS IN LIPSCHITZ FUNCTION SPACES

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ABSTRACT. We give a new characterization of the set $\text{ext}(B_{X^\#})$ of all extreme points of the unit ball $B_{X^\#}$ in the Banach space $X^\#$ of all Lipschitz functions on a metric space X . This result is applied to get a total variation characterization of $\text{ext}(B_{X^\#})$ in the particular case when X is a convex subset of a Banach space.

Let $0 \in X$ be an arbitrarily chosen point of a metric space $X = (X, d)$ which consists of at least two distinct points. Following Lindenstrauss [3] denote by $X^\#$ the Banach space of all functions $f : X \rightarrow \mathcal{R}$ such that $f(0) = 0$ and

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\} < \infty.$$

In other words, the Banach space $X^\#$ consists of all real-valued Lipschitz functions defined on X , which are equal zero at the distinguished point 0 . In the following, we always assume that the distinguished point 0 is equal to the origin of the Banach space E , whenever X is a subset of E containing the origin of E .

In the study of geometric Banach space theory and its various applications it is important to have a good characterization of the extreme points of unit balls. The investigation of the set of all extreme points $\text{ext}(B_{X^\#})$ of the unit ball $B_{X^\#}$ of $X^\#$ has been originated by Rolewicz [4] who has proved the following theorem.

Theorem A. *Let f be a function in $[0, 1]^\#$ with $\|f\| = 1$. Then $f \in \text{ext}(B_{[0,1]^\#})$ if and only if $|f'(x)| = 1$ a.e. on $[0, 1]$.*

Moreover, he has shown in [5] that a similar result cannot hold for the space $X = [0, 1] \times [0, 1]$ with Euclidean metric. Next, Cobzas [1] has characterized the extreme points in $X^\#$ for a rather restricted class of metric spaces X . Recently, Farmer [2] has presented a new characterization of the set $\text{ext}(B_{X^\#})$ without any additional restrictions on X . More precisely, he proved the following theorem.

Theorem B. *Let X be a metric space, and let f be a function in $X^\#$ with the norm $\|f\| = 1$. Then $f \in \text{ext}(B_{X^\#})$ if and only if (i) $\epsilon_{x,y}^f = 0$ for all $x, y \in X$,*

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where

$$\epsilon_{x,y}^f = \inf \left\{ \epsilon > 0 : d(x_{i-1}, x_i) - \epsilon_i \leq |f(x_{i-1}) - f(x_i)| \ (i = 1, \dots, n), \right. \\ \left. x_0 = x, \ x_n = y, \ \sum_{i=1}^n \epsilon_i \leq \epsilon \right\}$$

with the infimum taken over all finite sequences $\epsilon_1, \dots, \epsilon_n > 0$ and $x_1, \dots, x_{n-1} \in X$ satisfying the above inequalities.

Moreover, he noted that condition (i) is equivalent to the condition

$$(ii) \ \epsilon_{x,0}^f = 0 \text{ for every } x \in X,$$

which is an immediate consequence of the triangle inequality

$$(1) \quad \epsilon_{x,y}^f \leq \epsilon_{x,z}^f + \epsilon_{z,y}^f; \ x, y, z \in X.$$

In this paper, we first apply Theorem B to derive a new characterization of $\text{ext}(B_{X^\#})$. Next, we use this result to obtain the following

Theorem 1. *Let X be a convex subset of a normed linear space $E = (E, \|\cdot\|)$, and let f be a function in $X^\#$ such that $\|f\| = 1$. Then $f \in \text{ext}(B_{X^\#})$ if and only if*

$$(i) \quad \inf \left\{ \sum_{i=1}^n \left(\|x_i - x_{i-1}\| - \int_0^1 |f'_{x_i, x_{i-1}}(t)| dt \right) : x_0 = x, \ x_n = y \right\} = 0$$

for all $x, y \in X$, where the infimum is taken over all finite sequences $x_1, \dots, x_{n-1} \in X$, and

$$f_{x_i, x_{i-1}}(t) = f((1-t)x_{i-1} + tx_i), \ 0 \leq t \leq 1.$$

For this purpose, let

$$\langle x, y \rangle = \{z \in X : d(x, y) = d(x, z) + d(z, y)\}$$

be the *metric interval* with endpoints $x, y \in X$. Additionally, let $(x_i)_0^n$ be a *metric subdivision* of $\langle x, y \rangle$ with $x \neq y$, i.e., let $x_0 = x$, $x_n = y$, $x_i \in \langle x, y \rangle$, $x_i \neq x_j$ for $i \neq j$, and

$$(2) \quad d(x, y) = \sum_{i=1}^n d(x_{i-1}, x_i).$$

Then we define

$$(3) \quad \rho_f(x, y) = \inf \left\{ \|f\| d(x, y) - \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \right\},$$

where the infimum is taken over all finite metric subdivisions $(x_i)_0^n$ of the interval $\langle x, y \rangle$. Additionally, we put $\rho_f(x, x) = 0$. Since points $x_0 = x$ and $x_1 = y$ form a subdivision of $\langle x, y \rangle$, it follows from (3) that

$$(4) \quad \rho_f(x, y) \leq \|f\| d(x, y) - |f(x) - f(y)|$$

for all $x, y \in X$. Further, we have

$$\rho_f(x, y) \leq \|f\| d(x, y) - \sum_{i=1}^{n+m} |f(x_i) - f(x_{i-1})|$$

for all metric subdivisions $(x_i)_0^n$ of $\langle x, z \rangle$ and $(x_i)_{n+1}^{n+m}$ of $\langle z, y \rangle$, where $z \in \langle x, y \rangle$. Hence one can take the first infimum over x_1, \dots, x_{n-1} and the second over $x_{n+2}, \dots, x_{n+m-1}$ to get

$$(5) \quad \rho_f(x, y) \leq \rho_f(x, z) + \rho_f(z, y), \quad z \in \langle x, y \rangle.$$

In general, ρ_f does not satisfy the triangle inequality. For example, let $\|x\|_p$ ($1 < p < \infty$) denote l^p - norm of $x = (x_1, x_2) \in X = \mathcal{R}^2$. Then we have

$$\rho_f(x, y) = \|x - y\|_p - |x_1 - y_1|$$

for the function $f(x) = x_1$. Hence we get

$$1 = \rho_f(x, y) > \rho_f(x, z) + \rho_f(z, y) = 2^{\frac{1}{p}} - 1,$$

whenever $x = (0, 0)$, $y = (0, 1)$ and $z = (1, 0)$.

In view of this example, we define

$$(6) \quad \sigma_f(x, y) = \inf \{ \rho_f(x, z_1) + \rho_f(z_1, z_2) + \dots + \rho_f(z_n, y) : z_1, \dots, z_n \in X, n \in \mathcal{N} \}$$

for all $x, y \in X$ and $f \in X^\#$. Clearly, σ_f is a symmetric function such that $\sigma_f(x, x) = 0$ and

$$(7) \quad 0 \leq \sigma_f \leq \rho_f.$$

In particular, this together with (4) gives

$$(8) \quad |f(x) - f(y)| \leq \|f\| d(x, y) - \sigma_f(x, y); \quad x, y \in X.$$

Further, taking the infimum over $(z_i)_1^n$ and $(y_i)_1^m$ of the right-hand side of the inequality

$$\begin{aligned} \sigma_f(x, y) \leq & [\rho_f(x, z_1) + \rho_f(z_1, z_2) + \dots + \rho_f(z_n, z)] \\ & + [\rho_f(z, y_1) + \rho_f(y_1, y_2) + \dots + \rho_f(y_m, y)], \end{aligned}$$

we derive

$$\sigma_f(x, y) \leq \sigma_f(x, z) + \sigma_f(z, y),$$

and therefore

$$(9) \quad |\sigma_f(x, y) - \sigma_f(x, z)| \leq \sigma_f(y, z)$$

for all $x, y, z \in X$. Note also that

$$(10) \quad \sigma_f \leq \mu \leq \rho_f \implies \sigma_f = \mu,$$

whenever the function $\mu : X \times X \rightarrow \mathcal{R}$ satisfies the triangle inequality on X . Indeed, note that

$$\begin{aligned} \rho_f(x, z_1) + \rho_f(z_1, z_2) + \dots + \rho_f(z_n, y) & \geq \mu(x, z_1) + \mu(z_1, z_2) \\ + \dots + \mu(z_n, y) & \geq \mu(x, y) \geq \sigma_f(x, y), \end{aligned}$$

and take the infimum over $(z_i)_1^n$ to get $\sigma_f = \mu$.

Theorem 2. *Let X be a metric space, and let f be a function in $X^\#$ with the norm $\|f\| = 1$. Then $f \in \text{ext}(B_{X^\#})$ if and only if*

- (i) $\sigma_f(x, y) = 0$ for all $x, y \in X$.

Proof. Suppose first that $\sigma_f(x, y) = 0$ for all $x, y \in X$. Moreover, take an arbitrary $\epsilon > \rho_f(x, y)$. Then it follows from (2) – (3) that there exists a metric subdivision $(x_i)_0^n$ of $\langle x, y \rangle$ for which

$$(11) \quad d(x, y) - \epsilon = \left(\sum_{i=1}^n d_i \right) - \epsilon < \sum_{i=1}^n c_i,$$

where

$$d_i = d(x_{i-1}, x_i) > 0 \text{ and } c_i = |f(x_i) - f(x_{i-1})|.$$

Since $\|f\| = 1$, we have $c_i \leq d_i$. Moreover, by (11) one can find n numbers e_i ($i = 1, \dots, n$) such that $0 \leq e_i < c_i$ (if $c_i > 0$), $e_i = 0$ (if $c_i = 0$), and

$$\left(\sum_{i=1}^n d_i \right) - \epsilon = \sum_{i=1}^n e_i.$$

Now denote $\epsilon_i = d_i - e_i$. Then we have $\epsilon_i > 0$, $\sum_{i=1}^n \epsilon_i = \epsilon$, and $c_i \geq e_i = d_i - \epsilon_i$, i.e.,

$$d(x_{i-1}, x_i) - \epsilon_i \leq |f(x_i) - f(x_{i-1})| \quad (i = 1, \dots, n).$$

Hence it follows from the definition of $\epsilon_{x,y}^f$ that $\epsilon_{x,y}^f \leq \epsilon$. Since $\epsilon > \rho_f(x, y)$ was arbitrary, we conclude that

$$0 = \sigma_f(x, y) \leq \epsilon_{x,y}^f \leq \rho_f(x, y)$$

for all $x, y \in X$. This in conjunction with (1) enables to apply (10) in order to get $\epsilon_{x,y}^f = \sigma_f(x, y) = 0$. Thus Theorem B yields $f \in \text{ext}(B_{X^\#})$, which completes the proof of necessity.

For the proof of sufficiency, suppose that there exist $f \in X^\#$ and $z \in X$ for which $\|f\| = 1$ and $Y = \{y : \sigma_f(z, y) > 0\} \neq \emptyset$. Then the triangle inequality and symmetry of σ_f yield

$$(12) \quad \sigma_f(x, y) = \sigma_f(z, y)$$

for all $x \in X \setminus Y$ and $y \in Y$. This together with (8) and (9) enables to repeat mutatis mutandis Farmer's proof [2] of sufficiency of Theorem B, with $\epsilon_{x,y}^f$ replaced by $\sigma_f(x, y)$, in order to show that $f \notin \text{ext}(B_{X^\#})$. \square

From now on, we will assume that X is a convex subset of a normed linear space $(E, \|\cdot\|)$. In this case, we define

$$(13) \quad \hat{\rho}_f(x, y) = \inf \left\{ \|f\| \|x - y\| - \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \right\},$$

where the infimum is taken only over all finite subdivisions $(x_i)_0^n$ of the form

$$x_i = (1 - t_i)x + t_i y \quad (0 = t_0 < t_1 < \dots < t_n = 1).$$

It is clear that (2) holds for these algebraic subdivisions of the algebraic interval

$$[x, y] = \{(1 - t)x + ty : 0 \leq t \leq 1\},$$

and that $[x, y] = \langle x, y \rangle$ and $\hat{\rho}_f(x, y) = \rho_f(x, y)$ for all $x, y \in X$, whenever E is a strictly convex space. In general, we have only $\rho_f \leq \hat{\rho}_f$.

If $\hat{\sigma}_f(x, y)$ is defined by formula (6) with ρ_f replaced by $\hat{\rho}_f$, then $\sigma_f \leq \hat{\sigma}_f$. By the same arguments as above, one can also prove that $\hat{\rho}_f$ and $\hat{\sigma}_f$ satisfy inequality

(4) and the triangle inequality, respectively. In particular, by using (2) and (4) we obtain

$$\hat{\sigma}_f(x, y) \leq \hat{\rho}_f(x_0, x_1) + \dots + \hat{\rho}_f(x_{n-1}, x_n) \leq \|f\| \|x - y\| - \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$$

for all metric subdivisions $(x_i)_0^n$ of $\langle x, y \rangle$. Hence we derive $\hat{\sigma}_f \leq \rho_f$. Therefore, one can apply (10) with $\mu = \hat{\sigma}_f$ in order to get $\hat{\sigma}_f = \sigma_f$.

Lemma 1. *Let X be a convex subset of a normed linear space $E = (E, \|\cdot\|)$, and let $f \in X^\#$. Then we have*

$$\hat{\rho}_f(x, y) = \|f\| \|x - y\| - V_0^1(f_{x,y}) = \|f\| \|x - y\| - \int_0^1 |f'_{x,y}(t)| dt,$$

where $V_0^1(f_{x,y})$ denotes the total variation of the function $f_{x,y}$ defined by

$$f_{x,y}(t) = f((1 - t)x + ty) \quad (0 \leq t \leq 1).$$

Proof. By (13) we obtain

$$\begin{aligned} (14) \quad \hat{\rho}_f(x, y) &= \|f\| \|x - y\| - \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \right\} \\ &= \|f\| \|x - y\| - V_0^1(f_{x,y}), \end{aligned}$$

where the supremum is taken over all finite algebraic subdivisions $(x_i)_0^n$ of $[x, y]$. Since $f \in X^\#$, we have

$$|f_{x,y}(t) - f_{x,y}(s)| \leq \|f\| \|x - y\| |t - s| \quad (0 \leq t, s \leq 1).$$

Hence the derivative $f'_{x,y}(t)$ exists almost everywhere on $[0, 1]$, and the function $t \rightarrow f'_{x,y}(t)$ is integrable. Moreover, we have

$$V_0^1(f_{x,y}) = \int_0^1 |f'_{x,y}(t)| dt.$$

This in conjunction with (14) completes the proof. □

In view of the fact that $\hat{\sigma}_f = \sigma_f$, Theorem 1 is an immediate consequence of Lemma 1 and Theorem 2. Moreover, it follows from the triangle inequality for σ_f that Theorems 1 and 2 remain true, whenever we put either $x = 0$ or $y = 0$ into them. In particular, if the interval $X = [0, 1]$ is equipped with the metric $d(x, y) = |x - y|$, then Lemma 1 yields

$$\rho_f(x, y) = \hat{\rho}_f(x, y) = |x - y| \left(1 - \int_0^1 |f'(s)| ds \right)$$

for all $x, y \in [0, 1]$. On the other hand, by (5) and (7) one can apply (10) with $\mu = \rho_f$ to get $\sigma_f = \rho_f$. Hence Theorem A follows directly from Theorem 2.

Finally, we present another application of Theorem 2 which shows that the set $\text{ext}(B_{X^\#})$ of all extreme points of the unit ball $B_{X^\#}$ of $X^\#$ is quite rich, whenever

X is a normed linear space. For this purpose, denote by X^* the dual space of X , and note that

$$(15) \quad \sigma_f(x, x + \alpha z) = \sigma_f(0, \alpha z) = |\alpha| \sigma_f(0, z) \quad (\alpha \in \mathcal{R}; x, z \in X)$$

for every functional $f \in X^*$. To prove these identities, we need only to change variables $z_k \rightarrow z_k + x$ ($z_k \rightarrow \alpha z_k$) in the definition of $\hat{\sigma}_f = \sigma_f$ applied to $y = x + \alpha z$ ($y = \alpha z$, respectively), and use the identity

$$\hat{\rho}_f(x, y) = \|f\| \|x - y\| - |f(x - y)| \quad (x, y \in X),$$

which is a direct consequence of Lemma 1 and linearity of f . Since σ_f satisfies the triangle inequality, it follows from (15) that

$$\sigma_f(0, z_1 + z_2) \leq \sigma_f(0, z_1) + \sigma_f(z_1, z_1 + z_2) = \sigma_f(0, z_1) + \sigma_f(0, z_2).$$

This in conjunction with (15) means that the function $z \rightarrow \sigma_f(0, z)$ ($z \in X$) is a seminorm on X .

Theorem 3. *Let X be a normed linear space. Then we have*

$$\text{ext}(B_{X^\#}) \cap X^* = \text{ext}(B_{X^*}).$$

Proof. In view of definition of extreme points, we directly have

$$\text{ext}(B_{X^\#}) \cap X^* \subset \text{ext}(B_{X^*}).$$

Conversely, let a functional $f \in X^*$ be such that $\|f\| = 1$ and $f \notin \text{ext}(B_{X^\#})$. We need only to prove that $f \notin \text{ext}(B_{X^*})$. By Theorem 2 the set

$$Y = \{y : \sigma_f(0, y) > 0\}$$

is nonempty. Moreover, it follows from (15) that the set $X \setminus Y$ is a linear subspace of X which, in view of (12), has the property

$$(16) \quad \sigma_f(x, y) = \sigma_f(0, y) \quad (x \in X \setminus Y, y \in Y).$$

Now take a point $y_0 \in Y$, and denote by X_0 the linear subspace spanned by y_0 and $X \setminus Y$. Next, define the linear functional g on X_0 by the formula

$$g(x + \alpha y_0) = \alpha \sigma_f(0, y_0) \quad (x \in X \setminus Y, \alpha \in \mathcal{R}).$$

Then it follows from (15) and (16) that

$$(17) \quad |g(x + \alpha y_0)| = \sigma_f(x, x + \alpha y_0) = \sigma_f(0, x + \alpha y_0) \quad (x \in X \setminus Y, \alpha \in \mathcal{R}),$$

whenever $x + \alpha y_0 \in Y$. Otherwise, if $x + \alpha y_0 \notin Y$ then $\alpha = 0$ and (17) is obvious. Since the function $z \rightarrow \sigma_f(0, z)$ is a seminorm on X and g satisfies condition (17) on X_0 , it follows from the Hahn-Banach theorem that the functional $g : X_0 \rightarrow \mathcal{R}$ has an extension to the whole space X , which satisfies the inequality

$$|g(z)| \leq \sigma_f(0, z), \quad z \in X.$$

Consequently, one can apply (4) and (7) to get

$$|g(z)| \leq \|z\| - |f(z)|, \quad z \in X.$$

Thus $g \in X^*$ and $f_k \in X^*$ ($k = 1, 2$), where functionals f_k are defined by

$$f_k(z) = f(z) + (-1)^k g(z).$$

Therefore, we obtain

$$|f_k(z)| \leq |f(z)| + |g(z)| \leq \|z\|$$

for every $z \in X$. Hence we have $\|f_k\| \leq 1$ ($k = 1, 2$) and $f_k(y_0) \neq f(y_0)$, which in conjunction with the identity $f = (f_1 + f_2)/2$ shows that $f \notin \text{ext}(B_{X^*})$. Thus the proof is completed. \square

REFERENCES

1. S. Cobzas, *Extreme points in Banach spaces of Lipschitz functions*, *Mathematica* **31** (54) (1989), 25–33. MR **91j**:46032
2. J.D. Farmer, *Extreme points of the unit ball of the space of Lipschitz functions*, *Proc. Amer. Math. Soc.* **121** (1994), 807–813. MR **94i**:46038
3. J. Lindenstrauss, *On nonlinear projections in Banach spaces*, *Michigan Math. J.* **11** (1964), 263–287. MR **29**:5088
4. S. Rolewicz, *On extremal points of the unit ball in the Banach space of Lipschitz continuous functions*, *J. Austral. Math. Soc. Ser. A* **41** (1986), 95–98. MR **87k**:46022
5. S. Rolewicz, *On optimal observability of Lipschitz systems*, *Selected Topics in Operations Research and Mathematical Economics*, *Lecture Notes in Econom. and Math. Systems*, vol. 226, Springer-Verlag, Berlin, 1984, pp. 151–158. MR **86a**:49076

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