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ENTIRE SOLUTIONS OF FIRST-ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

JILL E. HEMMATI

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Abstract. We show that any entire solution of an essentially nonlinear first-order partial differential equation in two variables must be linear.

In this paper we consider complex-analytic solutions to some nonlinear first-order partial differential equations. Let $F(p,q)$ be an entire function in p,q. Suppose the zero set of F contains a complex line, i.e., $F(p,q) = (p + aq + b)S(p,q)$, where $a, b \in \mathbf{C}$ and S is an entire function. In this case, the partial differential equation $F(u_x, u_y) = 0$, $(x,y) \in \mathbf{C}^2$, has many entire solutions in $\mathbf{C}^2$, for example,

$$u(x,y) = -\frac{b}{2}x - \frac{b}{2a}y + f\left(x - \frac{y}{a}\right),$$

where f is any entire function of one variable. However, if $F(p,q)$ does not have a linear factor, then the entire solutions in $\mathbf{C}^2$ are completely characterized by the following:

Theorem. *Let u be an entire solution in $\mathbf{C}^2$ of $F(u_x, u_y) = 0$, where F is an entire function whose zero set $\{(p,q) \in \mathbf{C}^2 : F(p,q) = 0\}$ does not contain any complex lines, i.e., F does not have a linear factor. Then $u(x,y)$ is a linear function.*

Proof. Let $p = u_x$, $q = u_y$, and $z = u(x,y)$. Assume that u is not linear, i.e. u_x and u_y are not both constant. According to a corollary of the Weierstrass preparation theorem, we can factor F into irreducible (nonlinear) factors, since the ring of germs of holomorphic functions is a unique factorization domain (see [6]). Thus, we may assume that F is irreducible, and we can find $(x_0, y_0) \in \mathbf{C}^2$ such that $\operatorname{grad} F(p(x,y), q(x,y)) \neq 0$ while $F(p(x,y), q(x,y)) = 0$, for (x,y) near (x_0, y_0). Without loss of generality, we can assume that $F_q(p(x,y), q(x,y)) \neq 0$ near (x_0, y_0). The Hamilton-Jacobi equations for the characteristics (cf. [3]) are

$$\frac{dx}{dt} = F_p(p,q), \quad \frac{dy}{dt} = F_q(p,q), \quad \frac{dz}{dt} = pF_p(p,q) + qF_q(p,q), \quad \frac{dp}{dt} = \frac{dq}{dt} = 0.$$

By taking the initial curve $\Gamma : x(s, y_0) = s, y(s, y_0) = y_0$, with data $z(s, y_0) = f(s)$, where $f(s)$ is an entire function, we can complete it into a characteristic strip by choosing $p(s, y_0) = f'(s)$ and $q(s, y_0) = g(f'(s))$, where g solves $F(p, g(p)) = 0$.

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Then the characteristics with initial elements on Γ are given by

$$\begin{aligned} &x(s,t) = F_p(f'(s), g(f'(s)))t + s,\\ (1)\qquad &y(s,t) = F_q(f'(s), g(f'(s)))t + y_0,\\ &z(s,t) = [f'(s)F_p(f'(s), g(f'(s))) + g(f'(s))F_q(f'(s), g(f'(s)))]t + f(s). \end{aligned}$$

Thus, the characteristics are complex lines with slope $\frac{F_p}{F_q}(f'(s), g(f'(s)))$. Since F does not have a linear factor, the implicit function $g(p)$ does not have constant derivative. Thus, $\frac{F_p}{F_q}(f'(s), g(f'(s)))$, which is an analytic function of s, is not constant near $s = x_0$, provided $f'(s)$ is not constant. In this case, we must have

$$\frac{F_p}{F_q}(f'(s_1), g(f'(s_1))) \neq \frac{F_p}{F_q}(f'(s_2), g(f'(s_2))), \qquad f'(s_1) \neq f'(s_2) \text{ for some } s_1, s_2.$$

This would imply that the two characteristics passing through $(s_1, y_0), (s_2, y_0)$ intersect, and hence at some point $(x_1, y_1), u_x(x_1, y_1) = p(x_1(s,t), y_1(s,t)) = f'(s_1)$ and $u_x(x_1, y_1) = f'(s_2)$. This is impossible since u_x is an entire function, and hence must be single valued. Therefore, $f'(s) \equiv$ constant, $f(s)$ is linear, and by (1) we have that $u(x,y)$ is linear, which contradicts our assumption.

Remarks. (1) Our argument can be extended to include any holomorphic function F that does not have any linear pseudoprime factors.

(2) An example of an equation of this type is the eiconal equation in two variables

$$u_x^2 + u_y^2 - 1 = 0.$$

This particular case was treated in [5].

(3) The theorem fails in higher dimensions. Indeed, consider (cf. [5]) $u(x,y,z) = z + f(x+y)$ which solves $u_x^2 - u_y^2 + u_z - 1 = 0$. However, for some equations, for example the eiconal equation, the theorem stays true in all dimensions if one only considers real-valued solutions (see [5, 7, 12]).

(4) The following noteworthy corollary was communicated to us by Professor P. Ebenfelt.

Corollary. *Let $u(x,y)$ be a nonlinear entire function in $\mathbf{C}^2$. If the image of $\mathbf{C}^2$ under the gradient map $\nabla : (x,y) \mapsto (u_x(x,y), u_y(x,y))$ lies in an irreducible algebraic variety V, then V must be a complex line.*

(5) Our theorem seems to be very close in flavor to the celebrated theorem of S. Bernstein ([1, 2, 4, 8, 9, 10, 11]) that an entire solution of the minimal surface equation in two variables must be linear. It seems worthwhile to pursue this connection further.

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References

1. S. Bernstein, *Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus*, Math. Z. **26** (1927), 551–558.
2. L. Bers, *Isolated singularities of minimal surfaces*, Ann. of Math. (2) **53** (1951), 364–380. MR **13:**244c
3. F. John, *Partial Differential Equations*, 4th ed., Springer-Verlag, New York, 1982. MR **80f:**35001 (3rd ed.)
4. K. Jörgens, *Über die Lösungen der Differentialgleichung $rt - s^2 = 1$*, Math. Ann. **127** (1954), 130–134. MR **15:**961e

5. D. Khavinson, *A note on entire solutions of the eiconal equation*, Amer. Math. Monthly **102** (1995), 159–161. MR **95j:**35132
6. S. G. Krantz, *Function Theory of Several Complex Variables*, Wiley, 1982. MR **84c:**32001
7. G. Letac and J. Pradines, *Seules les affinités préservent les lois normales*, C. R. Acad. Sci. Paris Sér. A **286** (1978), 399–402. MR **57:**14100
8. E. J. Mickle, *A remark on a theorem of Serge Bernstein*, Proc. Amer. Math. Soc. **1** (1950), 86–89. MR **12:**13f
9. J. C. C. Nitsche, *Elementary proof of Bernstein's theorem on minimal surfaces*, Ann. of Math. (2) **66** (1957), 593-594. MR **19:**878f
10. J. C. C. Nitsche, *Lectures on Minimal Surfaces.* Vol. 1, Cambridge University Press, 1989. MR **90m:**49031
11. T. Rado, *Zu einem Satze von S.Bernstein über Minimalflächen im Gromen*, Math. Z. **26** (1927), 559–565.
12. O. N. Stavroudis and R. C. Fronczek, *Caustic surfaces and the structure of the geometrical image*, J. Opt. Soc. Amer. **66** (1976), 795–800. MR **54:**11984

Department of Mathematics, University of Arkansas, Fayetteville, Arkansas 72701