Entire Solutions of First-Order Nonlinear Partial Differential Equations

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Abstract. We show that any entire solution of an essentially nonlinear first-order partial differential equation in two variables must be linear.

In this paper we consider complex-analytic solutions to some nonlinear first-order partial differential equations. Let \( F(p, q) \) be an entire function in \( p, q \). Suppose the zero set of \( F \) contains a complex line, i.e., \( F(p, q) = (p + aq + b)S(p, q) \), where \( a, b \in \mathbb{C} \) and \( S \) is an entire function. In this case, the partial differential equation \( F(u_x, u_y) = 0 \), \((x, y) \in \mathbb{C}^2 \), has many entire solutions in \( \mathbb{C}^2 \), for example,

\[
  u(x, y) = -\frac{b}{2}x - \frac{b}{2a}y + f\left(x - \frac{y}{a}\right),
\]

where \( f \) is any entire function of one variable. However, if \( F(p, q) \) does not have a linear factor, then the entire solutions in \( \mathbb{C}^2 \) are completely characterized by the following:

**Theorem.** Let \( u \) be an entire solution in \( \mathbb{C}^2 \) of \( F(u_x, u_y) = 0 \), where \( F \) is an entire function whose zero set \( \{(p, q) \in \mathbb{C}^2 : F(p, q) = 0\} \) does not contain any complex lines, i.e., \( F \) does not have a linear factor. Then \( u(x, y) \) is a linear function.

**Proof.** Let \( p = u_x \), \( q = u_y \), and \( z = u(x, y) \). Assume that \( u \) is not linear, i.e. \( u_x \) and \( u_y \) are not both constant. According to a corollary of the Weierstrass preparation theorem, we can factor \( F \) into irreducible (nonlinear) factors, since the ring of germs of holomorphic functions is a unique factorization domain (see [6]). Thus, we may assume that \( F \) is irreducible, and we can find \((x_0, y_0) \in \mathbb{C}^2 \) such that \( \text{grad} \ F(p(x, y), q(x, y)) \neq 0 \) while \( F(p(x, y), q(x, y)) = 0 \), for \((x, y) \) near \((x_0, y_0) \). Without loss of generality, we can assume that \( F_q(p(x, y), q(x, y)) \neq 0 \) near \((x_0, y_0) \). The Hamilton-Jacobi equations for the characteristics (cf. [3]) are

\[
  \frac{dx}{dt} = F_p(p, q), \quad \frac{dy}{dt} = F_q(p, q), \quad \frac{dz}{dt} = pF_p(p, q) + qF_q(p, q), \quad \frac{dp}{dt} = \frac{dq}{dt} = 0.
\]

By taking the initial curve \( \Gamma : x(s, y_0) = s, y(s, y_0) = y_0 \), with data \( z(s, y_0) = f(s) \), where \( f(s) \) is an entire function, we can complete it into a characteristic strip by choosing \( p(s, y_0) = f'(s) \) and \( q(s, y_0) = g(f'(s)) \), where \( g \) solves \( F(p, g(p)) = 0 \).
Then the characteristics with initial elements on \( \Gamma \) are given by
\[
x(s, t) = F_p(f'(s), g(f'(s)))t + s, \\
y(s, t) = F_q(f'(s), g(f'(s)))t + y_0, \\
z(s, t) = [f'(s)F_p(f'(s), g(f'(s))) + g(f'(s))F_q(f'(s), g(f'(s)))]t + f(s).
\]
(1) Thus, the characteristics are complex lines with slope \( F \) does not have a linear factor, the implicit function \( g(p) \) does not have constant derivative. Thus, \( \frac{F_p}{F_q}(f'(s), g(f'(s))) \), which is an analytic function of \( s \), is not constant near \( s = x_0 \), provided \( f'(s) \) is not constant. In this case, we must have
\[
\frac{F_p}{F_q}(f'(s_1), g(f'(s_1))) \neq \frac{F_p}{F_q}(f'(s_2), g(f'(s_2))), \quad f'(s_1) \neq f'(s_2) \text{ for some } s_1, s_2.
\]
This would imply that the two characteristics passing through \((s_1, y_0), (s_2, y_0)\) intersect, and hence at some point \((x_1, y_1), u_x(x_1, y_1) = p(x_1(s, t), y_1(s, t)) = f'(s_1)\) and \(u_x(x_1, y_1) = f'(s_2)\). This is impossible since \(u_x\) is an entire function, and hence must be single valued. Therefore, \(f'(s) \equiv \text{constant}, f(s) \text{ is linear, and by (1) we have that } u(x, y) \text{ is linear, which contradicts our assumption.} \)

**Remarks.** (1) Our argument can be extended to include any holomorphic function \( F \) that does not have any linear pseudoprime factors.

(2) An example of an equation of this type is the eiconal equation in two variables
\[
u_x^2 + u_y^2 - 1 = 0.
\]
This particular case was treated in [5].

(3) The theorem fails in higher dimensions. Indeed, consider (cf. [5]) \( u(x, y, z) = z + f(x + y) \) which solves \( u_x^2 - u_y^2 + u_z - 1 = 0 \). However, for some equations, for example the eiconal equation, the theorem stays true in all dimensions if one only considers real-valued solutions (see [5, 7, 12]).

(4) The following noteworthy corollary was communicated to us by Professor P. Ebenfelt.

**Corollary.** Let \( u(x, y) \) be a nonlinear entire function in \( \mathbb{C}^2 \). If the image of \( \mathbb{C}^2 \) under the gradient map \( \nabla : (x, y) \mapsto (u_x(x, y), u_y(x, y)) \) lies in an irreducible algebraic variety \( V \), then \( V \) must be a complex line.

(5) Our theorem seems to be very close in flavor to the celebrated theorem of S. Bernstein ([1, 2, 4, 8, 9, 10, 11]) that an entire solution of the minimal surface equation in two variables must be linear. It seems worthwhile to pursue this connection further.

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**References**


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