

ON CARATHÉODORY'S CONDITIONS FOR THE INITIAL VALUE PROBLEM

D. C. BILES AND P. A. BINDING

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ABSTRACT. We prove a local existence theorem of Carathéodory–Goodman type for $\dot{x}(t) = f(t, x(t))$ where instead of $f(t, \alpha)$ being continuous in α we require only that it have no “downward discontinuities.”

1. INTRODUCTION

Let f be a real valued function defined on a neighbourhood of the point (t_0, x_0) in the (t, α) plane. Carathéodory's name is usually associated with the three conditions:

- (C1) (continuity) for almost all t , $f(t, \alpha)$ is continuous in α ;
- (C2) (measurability) for each α , $f(t, \alpha)$ is (Lebesgue) measurable in t ;
- (C3) (boundedness) for some absolutely continuous (AC) function m , $|f(t, \alpha)| \leq \dot{m}(t)$ for each α and almost all t .

Indeed, in 1918 Carathéodory [6] established a local existence result under (C1–C3) for AC solutions x of the initial value problem

$$(1) \quad \dot{x} = fx$$

given $x(t_0) = x_0$, where $fx : t \rightarrow f(t, x(t))$, and the equality in (1) is to be understood in the L_1 sense. In 1970, Goodman [7] improved this to

Theorem 1. *Assume (C1–C3), and let $x(t) = \sup\{y(t) : y \in M, \dot{y} \leq fy\}$ where the inequality is to be understood in the L_1 sense. Then x is a solution of (1).*

(Here M is the set of AC functions y for which $y(t_0) = x_0$ and $|\dot{y}| \leq \dot{m}$. Actually Goodman used all AC functions z satisfying $z(t_0) \leq x_0$ for his subfunctions, but M simplifies things a little). As Goodman notes, Theorem 1 includes the comparison theorem for differential inequalities, and characterizes the (pointwise) maximal solution for (1).

Our aim is to improve Theorem 1 by weakening (C1). Specifically, we follow Tarski [14] and make

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Definition 1. A real valued function g defined on a real open interval A is quasi-increasing ($q \uparrow$) if for each $\alpha \in A$,

$$(2) \quad \overline{\lim}_{\gamma \uparrow \alpha} g(\gamma) \leq g(\alpha) \leq \underline{\lim}_{\gamma \downarrow \alpha} g(\gamma)$$

and we modify (C1) and (C2) as follows:

- (C1') For almost all t , $f(t, \alpha)$ is $q \uparrow$ in α .
- (C2') For each AC function x , fx is (Lebesgue) measurable.

Section 2 is devoted to the proof of

Theorem 1'. *Theorem 1 holds with (C1) and (C2) replaced by (C1') and (C2').*

In Section 3 we compare our hypotheses and methods with others in the literature, but we note here that (C1) and (C2) (and most other conditions in the literature) automatically imply (C2'). Moreover (C1') includes not only (C1) but also the condition

$$(C\uparrow) \quad f(t, \alpha) \text{ is nondecreasing in } \alpha,$$

which has been investigated by several authors. Finally (C1') and (C2') include all modifications of (C1) and (C2) we have seen that apply in an unrestricted neighbourhood of (t, α) . Restricting the neighbourhood, for example to a cone as in [5], seems to require different methods, which we hope to consider elsewhere.

2. THE MAIN RESULT

We base the proof on

Lemma 1. *Assume (C1'), (C2') and (C3). Then there is a real valued function h defined on a neighbourhood of $(t_0, x_0, x_0) \in \mathbf{R}^3$ so that*

- (i) $h(t, \alpha, \alpha) = f(t, \alpha)$;
- (ii) for almost all t and each α , $h(t, \alpha, \beta)$ is nondecreasing in β ;
- (iii) for each $x \in M$,

$$(3) \quad k(t, \alpha) := h(t, \alpha, x(t))$$

satisfies (C1)–(C3) with f replaced by k .

Proof. Let

$$\begin{aligned} h(t, \alpha, \beta) &= \sup\{f(t, \delta) : \alpha \leq \delta \leq \beta\} \text{ if } \alpha \leq \beta, \\ &= \inf\{f(t, \delta) : \beta \leq \delta \leq \alpha\} \text{ if } \beta \leq \alpha. \end{aligned}$$

Then (i) and (ii) are clear, as is (C3) in (iii). To prove (C1) with f replaced by k we note that, for almost all t and each β , $h(t, \alpha, \beta)$ is nonincreasing in α , so it will suffice to prove that $h(t, \alpha, \beta)$ is $q \uparrow$ in α .

Suppose $\alpha \leq \beta$. Then

$$\begin{aligned} \overline{\lim}_{\gamma \uparrow \alpha} h(t, \gamma, \beta) &= \overline{\lim}_{\gamma \uparrow \alpha} \sup\{f(t, \delta) : \gamma \leq \delta \leq \beta\} \text{ since } \gamma \leq \beta \\ &= \overline{\lim}_{\gamma \uparrow \alpha} \sup\{\sup\{f(t, \delta) : \gamma \leq \delta < \alpha\}, \sup\{f(t, \delta) : \alpha \leq \delta \leq \beta\}\} \\ &\leq \sup\{f(t, \alpha), \sup\{f(t, \delta) : \alpha \leq \delta \leq \beta\}\} \text{ by (C1') for almost all } t \\ &= h(t, \alpha, \beta). \end{aligned}$$

Now suppose $\alpha > \beta$. Then

$$\begin{aligned} \overline{\lim}_{\gamma \uparrow \alpha} h(t, \gamma, \beta) &= \overline{\lim}_{\gamma \uparrow \alpha} \inf\{f(t, \delta) : \beta \leq \delta \leq \gamma\} \text{ when } \gamma \geq \beta \\ &\leq \inf\{f(t, \delta) : \beta \leq \delta \leq \alpha\} \text{ by (C1')} \text{ for almost all } t \\ &= h(t, \alpha, \beta). \end{aligned}$$

The proof of the other half of (2) for h is similar.

Finally, to prove (C2) with f replaced by k , we shall show that the set $\Gamma = \{t : k(t, \alpha) \leq \gamma\}$ is measurable for each real γ . Now

$$\begin{aligned} \Gamma &= (\Gamma \cap \{t : x(t) \leq \alpha\}) \cup (\Gamma \cap \{t : x(t) \geq \alpha\}) \\ &= \{t : \iota(t) \leq \gamma\} \cup \{t : \sigma(t) \leq \gamma\} \end{aligned}$$

where $\iota(t) = \inf\{f(t, \delta) : x(t) \leq \delta \leq \alpha\}$, $\sigma(t) = \sup\{f(t, \delta) : \alpha \leq \delta \leq x(t)\}$.

To prove that ι is measurable, we write $r(t) = \min\{x(t) + \rho, \alpha\}$ for any rational $\rho \geq 0$. Fix t so that (C1') holds, and let $\varepsilon > 0$ be given. Then $f(t, \gamma) < \iota(t) + \varepsilon$ for some $\gamma \in [x(t), \alpha]$. By (C1'),

$$fr(t) < f(t, \gamma) + \varepsilon \text{ for some rational } \rho.$$

It follows that $fr(t) < \iota(t) + 2\varepsilon$, so in fact

$$\iota(t) = \inf\{fr(t) : \rho \text{ is a nonnegative rational}\}.$$

Since r is AC for each such ρ , ι is the (essential) infimum of countably many functions which are measurable by (C2'), and hence ι is measurable. Similarly σ is measurable, so Γ is a measurable set. □

Remark. In the reverse direction, it is easily seen that if, for almost all t , $h(t, \alpha, \beta)$ is continuous in α and nondecreasing in β , then $f(t, \alpha) = h(t, \alpha, \alpha)$ satisfies (C1').

Proof of Theorem 1'. Without loss of generality we shall assume $m(t_0) = x_0$, so by (C3), $-m \in Y$, where $Y = \{y \in M : \dot{y} \leq fy \text{ a.e.}\}$. Thus $x(t) = \sup\{y(t) : y \in Y\}$ is defined (and finite). If $t < \tau$ and $\varepsilon > 0$ is given, there is $y \in Y$ with $y(\tau) \geq x(\tau) - \varepsilon$, so since $y(t) \leq x(t)$ and $y \in M$,

$$x(\tau) - x(t) \leq y(\tau) - y(t) + \varepsilon \leq \int_t^\tau \dot{m} + \varepsilon.$$

Similarly there is $z \in Y$ with $z(t) \geq x(t) - \varepsilon$, so since $z(\tau) \leq x(\tau)$ and $z \in M$,

$$x(\tau) - x(t) \geq z(\tau) - z(t) - \varepsilon \geq -\int_t^\tau \dot{m} - \varepsilon.$$

It follows that $x \in M$. In particular, x is AC. Thus the function k defined by (3) satisfies (C1)–(C3) with f replaced by k , by virtue of Lemma 1. In particular, if we write

$$Z = \{z \in M : \dot{z} \leq kz \text{ a.e.}\}$$

then by Theorem 1 the function s defined by $s(t) = \sup\{z(t) : z \in Z\}$ satisfies $\dot{s} = ks$.

For any $y \in Y$,

$$\dot{y}(t) \leq h(t, y(t), y(t)) \leq ky(t) \text{ for almost all } t$$

so $y \in Z$. This shows that $x \leq s$. But then

$$\dot{s}(t) = h(t, s(t), x(t)) \leq fs(t)$$

so $s \in Y$. This yields $s \leq x$, so in fact $s = x$ whence $\dot{x} = fx$. □

3. EXAMPLES AND REMARKS

Recall that the conditions (C1), (C1'), (C↑), etc., are defined in Section 1.

3.1. (C↑) seems to have been used first in connection with (Picard's) successive approximations which satisfy $\dot{x}_n = fx_{n-1}$. Nagumo [11] showed that x_n converge monotonically to a solution of (1) provided f is also continuous. (This result has been rediscovered several times.) Wend [15] derived the same conclusion from (C↑) and the assumption that $f(t, \alpha)$ is also nonnegative and nondecreasing in t (but with no continuity conditions). Muldowney/Willett [10] obtained a similar result from a slightly different iteration, via an abstract intermediate value theorem. When specialised to differential equations, their conditions require $f(t, \alpha)$ to be of the form $h(t, \alpha, \alpha)$ with h as in Lemma 1 and with an additional (semi) continuity condition in the third argument [10, Theorem 3.1]. Some condition extra to (C↑) is necessary for convergence of the x_n to a solution of (1), as is shown by an example in [3].

3.2. Another approach to the case when (C↑) holds is provided by Tarski's fixed point theorem [14]. An indication of the argument was given by Azbelev/Calyuk [1], with a more complete proof (though see 3.5) by Hanson/Waltman [9] a year later. Tarski actually proved an abstract intermediate value theorem for $q \uparrow$ functions g , noting ([14, p. 295, footnote]) that it continued to hold if the right-hand (say) inequality in (2) was replaced by

$$(4) \quad g(\alpha) \leq \overline{\lim}_{\gamma \uparrow \alpha} g(\gamma).$$

The following example satisfies this extension of $q \uparrow$, but (1) has no solution, thus showing that we cannot take Tarski's reasoning over directly to our problem.

Example 1. Let $f(t, \alpha) = (-1)^n$ if $(n+1)^{-1} \leq \alpha < n^{-1}$ for positive integers n and $f(t, 0) = 1, f(t, \alpha) = 0$ if $\alpha < 0$. We note that (2), modified by (4), is satisfied. On the other hand, (1) is autonomous for this example, and it fails the necessary condition for existence in [4, Theorem 3.5(ii)] if $t_0 = x_0 = 0$.

3.3. As noted, Goodman obtained the maximal solution from Theorem 1. For the case where (C↑) holds instead of (C1), Biles [3] obtained a similar result. The subfunction method was first used by Peano [12], who showed that if f was continuous and the inequality in Theorem 1 was strict, then the *minimal* solution for (1) resulted. The following example shows that Peano's result does not hold under our conditions.

Example 2. Let

$$\begin{aligned} f(t, \alpha) &= 2\alpha/t \text{ if } 0 \leq \alpha/t \leq 1, t > 0 \\ &= 2 \text{ if } \alpha > t \geq 0 \\ &= 0 \text{ if } \alpha \leq 0 \text{ or } t < 0. \end{aligned}$$

If $t_0 = x_0 = 0$, then the minimal and maximal solutions are given by $x(t) = 0$ and $2t$ respectively (since $0 \leq f(t, \alpha) \leq 2$ for all t and α). Let $0 < \varepsilon < 1$ and consider

$$\begin{aligned} x_\varepsilon(t) &= \varepsilon^{-1}t^2 + \varepsilon t && \text{for } 0 \leq t \leq \varepsilon - \varepsilon^2 \\ &= (2 - \varepsilon)t - \varepsilon(1 - \varepsilon)^2 && \text{for } \varepsilon - \varepsilon^2 \leq t. \end{aligned}$$

Then $\dot{x}_\varepsilon(t) = fx_\varepsilon - \varepsilon$ for all $t \geq 0$, and $x_\varepsilon(0) = 0$ so x_ε is a (strict) subfunction, but for each $t > 0$, $|x_\varepsilon(t) - 2t| < \varepsilon(t + 1)$ provided $\varepsilon < t$. It follows that the supremum of the (strict) subfunctions is the *maximal* solution to (1) in this case.

3.4. Explicit conditions along the lines of (C1') seem to start with Wu [16] who assumed that

$$\lim_{\gamma \uparrow \alpha} f(t, \gamma) \leq f(t, \alpha) = \lim_{\gamma \downarrow \alpha} f(t, \gamma).$$

Biles [2] improved this to

$$\overline{\lim}_{\gamma \uparrow \alpha} f(t, \gamma) \leq f(t, \alpha) = \lim_{\gamma \downarrow \alpha} f(t, \gamma),$$

and this in turn has recently been relaxed to

$$(5) \quad \overline{\lim}_{\gamma \uparrow \alpha} f(t, \gamma) \leq f(t, \alpha) = \underline{\lim}_{\gamma \downarrow \alpha} f(t, \gamma)$$

in [13]. These authors used rather complicated polygon-type methods of proof. Note that (C1') corresponds to \leq instead of $=$ in (5). A similar condition (in a sense a uniform version of (C1')) is used in [8]. The authors assume, for some $p \in L_1$,

$$f(t, \alpha) - f(t, \beta) \geq p(t)(\alpha - \beta) \quad \text{if } \alpha \geq \beta$$

and they prove a number of results including local existence.

3.5. Although (C2') is stronger than (C2), there is no difference for many conditions in the literature. Indeed, it is explicitly shown in [13] that (5) and (C2) imply (C2') (actually for any continuous x). Moreover, all conditions we know of that are not covered by (5) have been complemented by (C2') (or something stronger), with one exception. That exception is [9] where it is assumed that f is bounded and measurable on its domain, and satisfies (C↑). The following example (taken from [3]) satisfies those conditions (at least if Lebesgue measure is understood).

Example 3. Let S be a non-measurable subset of \mathbf{R} , and

$$\begin{aligned} f(t, \alpha) &= 1 && \text{if } \alpha \geq t \in S \quad \text{or} \quad \alpha > t \notin S \\ &= 0 && \text{otherwise.} \end{aligned}$$

On the other hand, fx is non-measurable if we choose $x(t) = t$ for all t . This violates a key hypothesis in the application of Tarski's theorem in [9]. One could, of course, understand measurability in a stronger (e.g. Borel) sense, but this analysis does raise the question of whether (C1'), (C2) and (C3) are sufficient for existence of solutions to (1).

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DEPARTMENT OF MATHEMATICS, WESTERN KENTUCKY UNIVERSITY, BOWLING GREEN, KENTUCKY 42101

E-mail address: Daniel.Biles@wku.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CALGARY, ALBERTA, CANADA T2N 1N4

E-mail address: binding@acs.ucalgary.ca