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CHARACTERIZATIONS OF PSEUDODIFFERENTIAL OPERATORS ON THE CIRCLE

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Abstract. Globally defined operators are shown to be equivalent to the classical pseudodifferential operators on the circle. A characterization of the smooth operators for the regular representation of $\mathbb{S}^1$ is also given.

1. Introduction

The introduction of a symbol into the Fourier inversion formula on the circle defines a "discrete" pseudodifferential operator, as in the case of the pseudodifferential operators on $\mathbb{R}$. Several classes of these globally defined operators have been shown to be equivalent to classical pseudodifferential operators on $\mathbb{S}^1$, the first (unpublished) result of this kind being attributed by M. S. Agranovich [2] to L. R. Volevich. Theorems on this theme are found, for example, in the papers [1], [2], [11] and [8], where also many applications are developed or mentioned.

The smooth vectors for the canonical action of the Heisenberg group on $\mathcal{L}(L^2(\mathbb{R}))$, the algebra of bounded operators on $L^2(\mathbb{R})$, were characterized by H. O. Cordes [5] as those pseudodifferential operators whose symbols have bounded derivatives of all orders. Similar criteria hold for other Lie groups [4, 5]. These results have been applied to nonlinear problems [9] and are also related to deformations of C^*-algebras [10].

In Section 2 of this paper, the discrete pseudodifferential operators are studied in detail. Theorem 2 is then a consequence of H. O. Cordes' smoothness theorem and of the equivalence between certain local pseudodifferential operators and a class of discrete pseudodifferential operators. The smooth vectors for the canonical action of the translation group on $\mathcal{L}(L^2(\mathbb{S}^1))$ are thus characterized as those discrete pseudodifferential operators whose symbols have all derivatives bounded in $\mathbb{S}^1$, uniformly with respect to the discrete variable. This means, in particular, that the class $Op\psi_d t_0$ defined in Section 3 is a Ψ^*-algebra in the sense of B. Gramsch [6]. E. Schrohe's results [12] imply, in particular, that the class $Op\psi_d c_0$ defined in Section 2 is also a Ψ^*-algebra.

The present results can be extended to the n-dimensional torus with a few minor changes of definitions and notation. For the sake of transparency of the argument, however, only the case of $n = 1$ is exposed. It must be pointed out that our

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Theorem 1 is not completely original (see W. McLean's Theorem 4.4 of [8]), but its proof is somewhat simpler.

2. DISCRETE PSEUDODIFFERENTIAL OPERATORS

The symbols considered here are of the form $b(x,j)$, $x \in \mathbb{R}$ and $j \in \mathbb{Z}$, with $b(\cdot,j)$ being a 2π-periodic smooth function for each j. Denoting by ∇^l the l-th power of the difference operator acting on sequences, $(\nabla a)_j = a_{j+1} - a_j$, and by $\langle\cdot\rangle$ the function $\langle t\rangle = \sqrt{1+t^2}$, $t \in \mathbb{R}$, we say that b is a *discrete classical symbol of order* m if, for every pair of non-negative integers (k,l), there is a constant C_{kl} such that

$$|\nabla^l \partial_x^k b(x,j)| \le C_{kl} \langle j\rangle^{m-l}, \ j \in \mathbb{Z}. \tag{1}$$

The smallest C_{kl} for which (1) holds is denoted by $\|b\|_{kl}$ and the class of all b satisfying (1) is denoted by $\psi_d c_m$.

A *discrete pseudodifferential operator* B acts on functions on the circle $\mathbb{S}^1$ and is defined by the formula

$$Bv(e^{ix}) = \sum_{j\in\mathbb{Z}} e^{ijx} b(x,j) v_j, \tag{2}$$

where v_j denotes the Fourier coefficient $v_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ijx} v(e^{ix}) dx$.

Proposition 1. *If b is a discrete classical symbol of order $m \in \mathbb{Z}$, then the equation* (2) *defines a continuous operator $B : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$. If $m \le 0$, then B extends to a bounded operator on $L^2(\mathbb{S}^1)$ whose norm satisfies*

$$\|B\| \le \max\{\|b\|_{2,0}, \|b\|_{0,0}\} \cdot \sum_{j\in\mathbb{Z}} \langle j\rangle^{-2}. \tag{3}$$

Proof. That B is a well-defined continuous operator on $C^\infty(\mathbb{S}^1)$ follows from

$$\sum_{j\in\mathbb{Z}} |j^l \partial_x^k b(x,j) v_j| \le \|b\|_{k,0} \cdot \sup|(1-\partial_x^2)^p v(e^{ix})| \cdot \sum_{j\in\mathbb{Z}} \langle j\rangle^{m+l-2p},$$

where, for each l, p is chosen so that $2p - m - l \ge 2$.

In order to prove the L^2-boundedness, one needs to estimate the matrix elements $b_{kj} = (2\pi)^{-1}\int_{-\pi}^{\pi} b(x,j) e^{i(j-k)x} dx$. Using that $e^{i(j-k)x} = \langle j-k\rangle^{-2}(1-\partial_x^2)e^{i(j-k)x}$ and integrating by parts, one gets $|b_{kj}| \le \max\{\|b\|_{2,0}, \|b\|_{0,0}\}\langle j-k\rangle^{-2}$. It follows that the sum $\sum_{j\in\mathbb{Z}} |b_{kj}|$ is bounded by the constant on the right-hand side of (3) for every k and that the corresponding statement with the roles of k and j interchanged is also true. The proposition follows from Schur's lemma (see, for example, [3]). ☐

The class of all operators B with discrete classical symbols of order m is denoted by $Op\psi_d c_m$. Next result states that it coincides with the usual class of pseudodifferential operators on the circle.

Theorem 1. *An operator $B : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ belongs to $Op\psi_d c_m$, $m \in \mathbb{Z}$, if and only if it is a pseudodifferential operator of order m on $\mathbb{S}^1$ in the usual sense; i.e., $B \in \Psi^m(\mathbb{S}^1)$ in the notation of* [7], *or $B \in LC_m(\mathbb{S}^1)$ in the notation of* [4].

Proof. Let $B \in Op\psi_d c_m$ be given. For each $\theta \in \mathbb{R}$, let us consider the chart $\chi : \mathbb{S}^1 \setminus \{-e^{i\theta}\} \to (-\pi+\theta, \pi+\theta)$, $\chi(e^{ix}) = x$. Given φ and ω in $C^\infty(\mathbb{S}^1)$, both vanishing on a neighborhood of $-e^{i\theta}$, let B_θ denote the operator on the Schwartz space $\mathcal{S}(\mathbb{R})$ defined by $(B_\theta u) \circ \chi = \omega B[(\varphi_\theta u) \circ \chi]$, where $\lambda_\theta = \lambda \circ \chi^{-1}$. In order to prove that $B \in \Psi^m(\mathbb{S}^1)$, it is enough to show that B_θ is a pseudodifferential operator on $\mathbb{R}$ whose symbol satisfies the estimates on (7).

The distribution kernel of B_θ as an operator on the Schwartz space $\mathcal{S}(\mathbb{R})$ equals

$$\frac{1}{2\pi}\sum_{j\in\mathbb{Z}} e^{i(x-y)j} b(x,j)\varphi_\theta(x)\omega_\theta(y),$$

where the sum converges in distribution sense. If B_θ is a pseudodifferential operator on $\mathbb{R}$, its symbol must then be given by (see [4], Chapter 1, for example)

$$a(x,\xi) = \frac{\varphi_\theta(x)}{2\pi}\sum_j b(x,j)\int e^{i(j-\xi)z}\omega_\theta(x-z)dz\,; \tag{4}$$

and conversely, if we show that this expression defines a proper symbol, then the equality $B_\theta = a(x,D)$ will be established.

The domain of the integral in (4) may be taken as the interval $[-2\pi+\delta, 2\pi-\delta]$, for some $\delta > 0$, because the supports of φ_θ and of ω_θ are both compact and contained in $(-\pi+\theta, \pi+\theta)$. The sum in equation (4) is absolutely convergent, since the integral appearing on the right-hand side is of the order of $\langle j-\xi\rangle^{-2p}$, for any integer p. The derivatives with respect to ξ may be brought inside the sum, since the absolute convergence also holds if we replace $\omega_\theta(x-z)$ by $z^l\omega_\theta(x-z)$, $l \geq 1$. This power of z and the use of the formula $e^{ijz} = (e^{iz}-1)^{-l}\nabla^l e^{ijz}$ allow us to do partial summation as follows:

$$\partial_\xi^l a(x,\xi) = \frac{\varphi_\theta(x)}{2\pi}\sum_j b(x,j)\nabla^l\left(\int_{-2\pi+\delta}^{2\pi-\delta} e^{i(j-\xi)z}\frac{(-iz)^l}{(e^{iz}-1)^l}\omega_\theta(x-z)dz\right) =$$

$$(-1)^l\frac{\varphi_\theta(x)}{2\pi}\sum_j[\nabla^l b(x,j-l)]\int_{-2\pi-\delta}^{2\pi+\delta} e^{i(j-\xi)z}\frac{(-iz)^l}{(e^{iz}-1)^l}\omega_\theta(x-z)dz.$$

Using that $e^{i(j-\xi)z} = \langle j-\xi\rangle^{-2p}(1-\partial_z^2)^p e^{i(j-\xi)z}$, integrating by parts and using also (1), it follows that

$$|\partial_\xi^l a(x,\xi)| \leq C\sum_j \langle j\rangle^{m-l}\langle j-\xi\rangle^{-2p}, \tag{5}$$

for any non-negative integer p, with C equal to

$$2\|b\|_{0,l}\sup_j\frac{\langle j-l\rangle^{m-l}}{\langle j\rangle^{m-l}}\sup_{|x-\theta|<\pi,|z|<2\pi-\delta}\left|\varphi_\theta(x)(1-\partial_z^2)^p\frac{z^l\omega_\theta(x-z)}{(e^{iz}-1)^l}\right|.$$

Using now that $\langle x\rangle \leq 2\langle y\rangle\langle x-y\rangle$, for all real x and y, it follows that

$$|\partial_\xi^l a(x,\xi)| \leq C2^{|m-l|}\langle\xi\rangle^{m-l}\sum_{j\in\mathbb{Z}}\langle j-\xi\rangle^{|m-l|-2p}. \tag{6}$$

If p is large enough, this last sum defines a continuous and periodic function of ξ and is then bounded by a constant.

The previous argument also works after (4) is differentiated with respect to x. It then follows that a is a classical symbol of order m, i.e., it satisfies the estimate

$$|\partial_x^k\partial_\xi^l a(x,\xi)| \leq C_{k,l}\langle\xi\rangle^{m-l}\,. \tag{7}$$

Conversely, let $B \in \Psi^m(\mathbb{S}^1)$ be given. By means of a suitable choice of a partition of unity, it can be seen that it is enough to show that $B^\dagger = \varphi(M)B\omega(M)$ is in

$Op\psi_d c_m$ for any pair (φ, ω) of functions in $C^\infty(\mathbb{S}^1)$ both vanishing on a neighborhood of some point $e^{-i\theta}$. We have denoted by $\zeta(M)$ the operator multiplication by ζ. Since $B^\dagger$ is continuous on $C^\infty(\mathbb{S}^1)$, if it belongs to $Op\psi_d c_m$ its discrete symbol must then be given by

$$b(x,j) = e^{-ijx} B^\dagger e_j(e^{ix}), \tag{8}$$

where $e_j(z) = z^j$, $z \in \mathbb{S}^1$; and conversely, if equation (8) defines a discrete symbol in $\psi_d c_m$, the corresponding operator in $Op\psi_d c_m$ must then be equal to $B^\dagger$. All that is left is to show that b defined on (8) indeed satisfies (1).

By hypothesis, $B^\dagger$ is locally given as a pseudodifferential operator on $\mathbb{R}$ with symbol a satisfying (7). It then follows that

$$b(x,j) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(x,\xi) \left(\int_{-\pi+\theta}^{\pi+\theta} e^{i(x-y)(\xi-j)} \rho(y) dy \right) d\xi \tag{9}$$

where $\rho \in C_0^\infty(-\pi+\theta, \pi+\theta)$ is equal to one on the support of ω_θ. Using that $\nabla^l e^{ijz} = e^{ijz}(e^{iz}-1)^l$ and that $e^{i(x-y)\xi} = i^{-l}(x-y)^{-l} \partial_\xi^l e^{i(x-y)\xi}$, we get:

$$\nabla^l b(x,j) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(x,\xi) \partial_\xi^l \left(\int_{-\pi+\theta}^{\pi+\theta} e^{i(x-y)(\xi-j)} \left[\frac{e^{i(y-x)}-1}{i(x-y)} \right]^l \rho(y) dy \right) d\xi .$$

In order to integrate by parts in ξ, we need the estimate

$$\left| \partial_\xi^q \int_{-\pi+\theta}^{\pi+\theta} e^{i(x-y)(\xi-j)} \left[\frac{e^{i(y-x)}-1}{i(x-y)} \right]^l \rho(y) dy \right| \leq 2\pi A_{p,l,q}(x) \langle \xi - j \rangle^{-2p} ,$$

which holds for any integers $p \geq 0$ and $0 \leq q \leq l$, with

$$A_{p,l,q}(x) = \sup_{|y-\theta|<\pi} \left| (1-\partial_y^2)^p [(e^{i(y-x)}-1)^l (x-y)^{q-l} \rho(y)] \right| .$$

If p is chosen so that $m - l - 2p \leq -2$, then it follows from (7) that the desired integration by parts is legitimate and that

$$|\nabla^l b(x,j)| \leq C_{0,l} A_{p,l,0}(x) \int_{-\infty}^{+\infty} \langle \xi \rangle^{m-l} \langle \xi - j \rangle^{-2p} d\xi . \tag{10}$$

Since $A_{p,l,0}$ depends continuously on x and $b(x,j)$ is periodic in x, the inequality on (10) also holds with a constant independent of x. Using once more the inequality $\langle \xi \rangle^{m-l} \leq 2^{|m-l|} \langle j \rangle^{m-l} \langle \xi - j \rangle^{|m-l|}$ and requiring that $-2p + |m-l| \leq -2$, it follows that the integral on the right-hand side of (10) is bounded by the constant $2^{|m-l|} \langle j \rangle^{m-l} \int \langle \xi \rangle^{-2} d\xi$. This proves (1) for $k = 0$.

When (9) is differentiated with respect to x, some of the derivatives will fall on $a(x,\xi)$ and some on $e^{i(x-y)(\xi-j)}$. Those on the symbol cause no trouble, since any derivative of a with respect to x also satisfies (7) with different constants. Those on the exponential may be transformed into derivatives with respect to y, since $\partial_x e^{i(x-y)(\xi-j)} = -\partial_y e^{i(x-y)(\xi-j)}$. An integration by parts then brings the derivatives to ρ and thus the same argument used for the case when $k = 0$ will finish the proof. ☐

3. Smoothness criterion

Let us denote by $\psi_d t_m$, $m \in \mathbb{Z}$, the class of the symbols $b(x,j)$ satisfying (1) possibly only for $l = 0$. Proposition 1 still holds for $b \in \psi_d t_m$, since only the finiteness of the seminorms $\| \cdot \|_{k,0}$, $k \geq 0$, is needed in the proof. Denoting by $Op\psi_d t_m$ the class of all corresponding B defined by (2), it is also clear that a weaker version of Theorem 1 is true; namely, that a continuous $B : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ belongs to $Op\psi_d t_m$ if and only if, for every real θ and every pair of smooth functions (φ, ω) vanishing on a neighborhood of $e^{-i\theta}$, the operator B_θ defined at the beginning of the proof of Theorem 1 is a pseudodifferential operator on $\mathbb{R}$ with symbol a satisfying

$$|\partial_x^k \partial_\xi^l a(x,\xi)| \leq C_{k,l} \langle \xi \rangle^m . \tag{11}$$

Since the support of a projects onto a compact subset of the x-axis, such a symbol belongs to $\psi t_{(m,-\infty)}$, as defined in [4].

Let T_z, $z \in \mathbb{R}$, denote the unitary operator on $L^2(\mathbb{S}^1)$ defined by $(T_z u)(e^{ix}) = u(e^{i(x-z)})$. The following is now a consequence of Theorem VIII.2.1 of [4].

Theorem 2. *Given B a bounded operator on $L^2(\mathbb{S}^1)$, the map*

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{L}(L^2(\mathbb{S}^1)), \\ z & \longmapsto & T_z^{-1} B T_z \end{array} \tag{12}$$

has operator-norm derivatives of all orders if and only if $B \in Op\psi_d t_0$.

Proof. If $B \in \mathcal{L}(L^2(\mathbb{S}^1))$ is such that the map on (12) is smooth, the same is true for $\varphi(M)B\omega(M)$, where φ and ω are smooth functions on $\mathbb{S}^1$. If both functions vanish on a neighborhood of some point, the operators $B(z) = T_z^{-1}\varphi(M)B\omega(M)T_z$ may be viewed as operators on $L^2(\mathbb{R})$, defining an operator-valued function which is norm-smooth with respect to z, for small $z \in \mathbb{R}$. We must show that conjugation of $B(z)$ by the multiplication operators e^{iyM} is also $\mathcal{L}(L^2(\mathbb{R}))$-smooth with respect to y. Let φ_1 and ω_1 be smooth functions of supports slightly larger than those of φ and ω and identically equal to one on the supports of φ and of ω, respectively. We then have

$$e^{iyM}B(0)e^{-iyM} = [e^{iyM}\varphi_1(M)]B(0)[e^{-iyM}\omega_1(M)],$$

which is clearly smooth with respect to y. Using the commutation relation between T_z and e^{iyM}, it follows that

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathcal{L}(L^2(\mathbb{R})), \\ (z,y) & \longmapsto & e^{-iyM}T_{-z}(\varphi(M)B\omega(M))T_z e^{iyM} \end{array}$$

is norm smooth. Cordes' theorem quoted above then implies that $\varphi(M)B\omega(M)$ is locally given as a pseudodifferential operator on $\mathbb{R}$ with symbol $a \in \psi t_{(0,0)}$. It follows that B is continuous on $C^\infty(\mathbb{S}^1)$ and thus that B is in $Op\psi_d t_0$.

Conversely, a given $B \in Op\psi_d t_0$ can be written as a sum of operators of the type $\varphi(M)B\omega(M)$, with φ and ω vanishing on a neighborhood of some point of the circle. Each of these operators is locally given as a pseudodifferential operator with symbol in $\psi t_{(0,0)}$. Again by Cordes' smoothness theorem, these local operators are smooth under the Heisenberg group action, in particular under the translation group action on $\mathcal{L}(L^2(\mathbb{R}))$. This shows that each of the operators $\varphi(M)B\omega(M)$ is $\mathcal{L}(L^2(\mathbb{S}^1))$-smooth under the action of the translation group of $\mathbb{S}^1$. $\square$

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