PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 5, May 1997, Pages 1545–1554
S 0002-9939(97)04132-4

SMALL INDUCTIVE DIMENSION OF COMPLETIONS OF METRIC SPACES

S. MRÓWKA

(Communicated by Franklin D. Tall)

Abstract. We construct a 0-dimensional metric space which under a special set-theoretic assumption, denoted in the paper as $\mathrm{S}(\aleph_0)$, does not have a 0-dimensional completion. Shortly after the submission of the paper for publication R. Dougherty has shown the consistency of $\mathrm{S}(\aleph_0)$. ($\mathrm{S}(\aleph_0)$ disagrees with the continuum hypothesis.)

1. Statement of the results

We shall be concerned here with what looks like one of the most innocent statements about metric spaces:

($\mathbf{Q}_0$). *Every metric space M having a base of clopen sets has a completion with such a base.*

(Q_0) can be treated as a statement in dimension theory. Agreeing that ind and dim stand for small inductive and covering dimensions, respectively, we shall define, for a metric space M, the *completion dimension*, $\mathrm{ind}_c M$, to be the $\min\{\mathrm{ind}\, M^* : M^* \text{ is a completion of } M\}$. We have $\mathrm{ind}_c\, M \leq \dim M$ (Katetov - Morita *ca* 1955).

Consider the statement

(Q). *For every metric space M,* $\mathrm{ind}_c\, M = \mathrm{ind}\, M$.

(Q_0) is the restriction of (Q) to the lowest case: $\mathrm{ind}\, M = 0$. (Q) is true if we insert "separable" before "metric".

One could speculate that our complete ignorance as to the validity of (Q) and (Q_0) is due to the fact that they are false or that, at least, they cannot be derived from the usual axioms of set theory. But the prospect of refuting (Q) seems quite dim indeed; one needs a very esoteric space – a metric space M with $\mathrm{ind}\, M < \dim M$. At present, only two such spaces are known and both of them sow horror among most mathematicians. And, regretfully, none of them is helpful here. The Roy's Δ is complete; the space μ_0 of [M1] is not complete, but it satisfies (Q_0).

In this paper we provide circumstantial evidence supporting these speculations. We shall construct a metric space $\nu\mu_0$ with $\mathrm{ind}\, \nu\mu_0 = 0$ and such that the condition $\mathrm{ind}_c\, \nu\mu_0 > 0$ is possibly consistent. In fact, it can be derived from the existence of subsets of the Cantor set with a certain property (denoted as (**P**) and formulated

---

Received by the editors November 20, 1995.

1991 *Mathematics Subject Classification.* Primary 54F45; Secondary 54A35, 54E35, 54H05.

Key words and phrases. Inductive and covering dimension, metric spaces, completion.

in sect. 4). Although this property belongs to a very thoroughly investigated area (borderline of set-theoretic topology and the theory of Borel and/or analytic sets), the existence of sets with this property seems to be a new and not instantly solvable problem. It disagrees with the continuum hypothesis, but, on the other hand, it does follow from "believably" consistent propositions of Set Theory.[1] The details of this discussion are given in sect. 5, at this time we wish to emphasize that we certainly do not bring the final solution of the problem – far from it – rather, we shift its location to where it becomes accessible to a wider audience.[2] But – depending upon the outcome – the continuation of these considerations will either settle (Q) in the negative or bring a consistency result or – in the worst case – we will have to end up with a displeasing conclusion that all that this paper contributes is just another esoteric space. And of course, the procedure presented here is not the only way to attack the problem; it is quite possible that someone blessed with the gray cells better than those at the disposal of the author[3] will come up with either further modification of $\nu\mu_0$ or an example based on an entirely new idea that would settle these matters without leading us to the pandemonium of undecidability.

Thus, although this paper does not give any definitive conclusion as to validity of (Q) and (Q_0), its publication is dictated by the extraordinary difficulty of this group of problems. Let us take a brief look at the history (comments that follow refer to *metric* spaces). Evidently, the progress was incredibly slow and, compared with what still has to be done, incredibly meager. Problems concerning dimension of non-separable metric spaces go back at least half a century; in particular, the one concerning the equality $\dim = \operatorname{ind}$ was explicitly stated by P. S. Alexandroff in 1935. The answer, Roy's Δ, was announced in 1962 [R1] (the full details were published only 6 years later [R2]). The forbidding complexity of Roy's construction manifests itself, e.g., in its resistance to any attempts of simplification. So far, only one book [Pe] did dare to reproduce Δ. Normally, a book presentation of a result is far better organized and frequently much shorter than the original one; in this case, it was on the level of a photographic copy. In fact, until [M1] (1972) and [M2] (presented in 1982, published in 1985) nobody was able to move even a single brick in Roy's edifice – but then the same become true about [M1] and [M2]. And Δ answers the question only on the lowest level: what it gives is that $\dim - \operatorname{ind}$ can be made equal to 1. Thirty years have already passsed and despite many lamentations we still have absolutely no idea how to answer the very next question: *Can the dimension spread* $\dim - \operatorname{ind}$ *be made greater than 1*? In [M2] it has been published that, ironically enough, the most obvious candidates – squares of Δ and μ_0 (in fact, loosely speaking, that squares of similarly built spaces) – cannot be used for this purpose.[4] [M2] contained several suggestions of possible ways of further attack. So far nothing has been done; indeed, so far [M2] seems to

---

[1] Unexpectedly, shortly after the submission of this paper for publication, R. Dougherty has shown the consistency of these propositions. In this paper we give only initial comments on this situation (sect. 8); a detailed study of the consequences of now known to be consistent assumptions will take same time.

[2] In fact, one of the problems generated in this way was already a subject of a Ph.D. dissertation of R. Dougherty; see sect. 5.

[3] The above should not be construed as the claim of non-emptiness of this set.

[4] See the comments in [M2] of contributions of J. Terasawa.

exceed the power of comprehension of anybody working in General Topology.[5] Note also that [M2] does not give a clean result – it assumes that $2^{\aleph_0}$, when measured through its position in the series of alephs, is not too far from $\aleph_0$.[6]

In regard to the completion dimension spread, $\operatorname{ind}_c - \operatorname{ind}$, the situation is even worse. We do not even know if $\operatorname{ind}_c - \operatorname{ind}$ can be made greater than 0 (i.e., we do not know if (Q_0) is false). This is the first paper which raises hope that progress can be made in this direction – but then, again, maybe only under special set-theoretic assumptions. Besides hope, this paper brings a general simplification of the matters. We introduce a new but very natural construction of metrizable uniformities; definition of $\nu\mu_0$ is an instance of this construction. Thus, in contrast to Δ and μ_0, $\nu\mu_0$ is no longer the result of a very artificial and isolated procedure. Although this construction cannot be directly applied to either Δ or μ, it should, by loose analogy, strip some mystery of the structure of these spaces. We thus hope that this paper will be accessible to a considerable number of workers in General Topology and will indeed induce them to join those who pursue these investigations. Joining such an exclusive (so far, one member) club is not easy, but one of the rewards of membership is the justifiable pride of being able to make the sentence "Survey of the dimension theory of non-separable metric spaces" no longer synonymous with "What S. Mrówka knows on this subject".

Concluding these lamentations, I wish to do more than just express my gratitude to the referee for his thorough job. In fact, I wish to express my appreciation for his mathematical power. Preparation of a paper involves the translation of a mental picture into a printable language. Such a translation frequently involves omissions of some facts which were completely obvious in the picture – the result is a mathematical text which is obviously false or simply does not make any sense. Besides vulgar misprints, the original version of this paper did contain such omissions, but the referee, with a perfect precision, always knew what is missing, where and why.

2. Construction of uniformities

For a member V of a uniformity (i.e., for a relation V) we will write $(x, y) \in V$ or xVy or $x \in V(y)$ interchangeably, whatever is typographically more convenient. $V \circ U$ stands for composition. A *uniform base* for a set is, of course, a base for a uniformity for this set.

Let $\mathcal{R} = \{R_n : n = 1, 2, \dots\}$ be a countable uniform base for a set T and, for every $t \in T$, let $\mathcal{S}^t = \{S^t_n : n = 1, 2, \dots\}$ be a countable uniform base for a set X. We say that the system of uniform bases $\mathcal{R}, \mathcal{S}^t$, $t \in T$, is *summable* provided that the following condition is satisfied:

(*) for every n and every t there exists an $m = \phi(n, t) > n$ so that for every $s \in T$, $s\, R_m\, t$ implies $S^s_m \subset S^t_n$,

---

[5] One of the obstacles in reading [M2] could be its very repulsive typography. [M2] is now typeset in $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX. There is even an extended version in which the number of phrases of the form "...verification of ... is routine...", "...obvious...", "...left to the reader..." have been reduced. Copies are available from the author on request.

[6] This possibly is an advantage being the first indication that these problems may depend upon set theoretic assumptions. [M2] assumes the continuum hypothesis but it also indicates that the constructions can probably be modified so that they can be carried over under the assumption $2^{\aleph_0} < \aleph_{\omega_0}$. There is now some evidence that, leaning a bit more heavily on Set Theory, the results of [M2] can be obtained under the assumption that $2^{\aleph_0}$ is Banach (i.e., real-valued) non-measurable. It is doubtful that $\dim \mu_0^2$ can be determined under the usual axioms of Set Theory.

and, for a summable system of uniform bases we define their *uniform sum*, $\mathcal{P} = \sum\{\mathcal{S}^t : t \in T\}$ to be the class of all P_n, $n = 1, 2, \dots,$ where P_n are defined in the following way: let $\psi(n,t)$ be the"shifted" iteration of $\phi(n,t)$: $\psi(0,t) = \phi(0,t)$, $\psi(n+1,t) = \phi(\psi(n,t)+2,t)+1$; we let

$$(x,t)\, P_n\, (y,s) \equiv \text{ there is an } (r,v) \text{ so that } \quad t\, R_{\psi(n,v)}\, v,\; v\, R_{\psi(n,v)}\, s,\; x\, S^v_{\psi(n,v)}\, r,\; r\, S^v_{\psi(n,v)}\, y, \tag{2.1}$$

We have:

2.1. *$\mathcal{P}$ is a uniform base for $X \times T$.*

Proof. Only the triangle inequality $P_{n+1} \circ P_{n+1} \subset P_n$ needs verification.

Let $(x,t)\, P_{n+1}\, (z,u)$ and $(z,u)\, P_{n+1}\, (y,s)$. There exist (r_0, v_0) and (r_1, v_1) so that, letting $m^i_j = \psi(n+i, v_j)$, $i, j = 0, 1$, we have

$$(t, v_0) \in R_{m^1_0},\; (v_0, u) \in R_{m^1_0},\; (x, r_0) \in S^{v_0}_{m^1_0},\; (r_0, z) \in S^{v_0}_{m^1_0},$$

and

$$(u, v_1) \in R_{m^1_1},\; (v_1, s) \in R_{m^1_1},\; (z, r_1) \in S^{v_1}_{m^1_1},\; (r_1, y) \in S^{v_1}_{m^1_1},$$

and, of course, $m^1_j = \phi(m^0_j + 2, v_j) + 1$. Assume that $m^1_0 \geq m^1_1$. We then have $(v_0, u) \in R_{m^1_0} \subset R_{m^1_1}$. Consequently,

$$(v_0, v_1) \in R_{\phi(m^0_1+2, v_1)}. \tag{2.2}$$

Since $m^0_1 + 2 < \phi(m^0_1 + 2, v_1)$, we have (t, v_1), $(v_1, s) \in R_{m^0_1}$. Furthermore, (2.2) implies that $S^{v_0}_{m^1_0} \subset S^{v_1}_{m^0_1+2}$, therefore (x, r_0), $(r_0, z) \in S^{v_1}_{m^0_1+2}$, thus $(x, z) \in S^{v_1}_{m^0_1+1}$ and further $(x, r_1) \in S^{v_1}_{m^0_1}$, but plainly $(r_1, y) \in S^{v_1}_{m^0_1}$ and (2.1) is satisfied for $(r, v) = (r_1, v_1)$. This ends the proof. ☐

We emphasize that the uniformity in $X \times T$ generated by the base $\mathcal{P}$ as well as the very condition (*) depends upon the choice of the particular bases $\mathcal{R}$ and $\mathcal{S}^t$. However, as far as the topologies are concerned the situation is more stable:

2.2. *If, for every $x \in X$ and every $t \in T$, $W^t_n(x)$ and $I_n(t,x)$ are descending local bases of x and t, respectively (relative to the topologies corresponding to the uniformities $\mathcal{S}^t$), and $\mathcal{R}$, then the sets $U_n(t,x) = W^t_n(x) \times I_n(t,x)$ form a neighborhood system for the topology in $X \times T$ corresponding to the uniformity generated by the base $\mathcal{P}$.*

Consequently, the topology in $X \times T$ depends only upon the topologies corresponding to $\mathcal{R}$ and $\mathcal{S}^t$.

Proof. Given an n, select an n_1 so that $(R_{n_1} \circ R_{n_1})(t) \subset I_n(t,x)$ and $(S^t_{n_1} \circ S^t_{n_1})(x) \subset W^t_n(x)$. Then $P_{n_1}(t,x) \subset W^t_n(x) \times I_n(t,x)$. Conversely, letting $m = \psi(n,t)$, select an n_1 with $I_{n_1}(t,x) \subset R_m(t)$ and $W^t_{n_1}(x) \subset S^t_m(x)$. Then $W^t_{n_1}(x) \times I_{n_1}(t,x) \subset P_n(t,x)$. ☐

2.2a. *Note.* In 2.2 we allow $I_n(t,x)$ to depend upon x; in other words, we allow the selection of different local bases of the same t. This is needed *only* in sect. 6. In other words, to get everything before sect. 6, one can simply select, for every t, just one local base $I_n(t)$. The reader will be reminded about this in relevant places.

For a given $x \in X$, let L_x be the set of all $(x,t) \in X \times T$.

2.3. *For a fixed x the function $h(x,t) = t$ maps L_x homeomorphically into T.*

Proof. h takes $U_n(x,t) \cap L_x$ onto $I_n(t,x) \cap h[L_x]$. $\square$

3. Definition of $\nu\mu_0$

Let t be a number from $I = [0,1]$. If t can be written in the form $t = (2k+1)2^{-n}$, where k is an integer and n is a non-negative integer, then (t is called a dyadic rational and) we let $\operatorname{ord} t = n$; if t is not a dyadic rational, then we let $\operatorname{ord} t = \infty$. Let $\mathfrak{C}$ be the Cantor set. Elements of $\mathfrak{C}$ will be treated as infinite sequences of 0's and 1's; they will be denoted by the letters $x, y, z, \ldots$ and $x|n = y|n$ will mean that the first n terms of x and y are equal. Infinite sequences of elements of $\mathfrak{C}$ will be denoted as $\tilde{x}, \tilde{y}, \tilde{z}, \ldots$ and furthermore $\tilde{x}(n)$ (and not $\tilde{x}_n$) will denote the n-th term of $\tilde{x}$ (thus, e.g., $\tilde{x}_1$, $\tilde{x}_2$ will denote two sequences).

(We recall that if $\mathcal{S}^k = \{S^k_n \colon n = 1, 2, \ldots\}$ are uniform bases, each for a set X^k, then by their product $\mathbb{P}_k \mathcal{S}^k$ we mean the class of all sets $S_n = \{(x,y) \in \mathbb{P}_k X^k \colon (x^i, y^i) \in S^i_n \text{ for every } i \leq n\}$, where (x^i, y^i) is the pair of the i-th coordinates of x and y, respectively.)

Now, in $\mathfrak{C}$ we will take $\mathcal{K}_0$ to be the uniform base generating the discrete uniformity and $\mathcal{K}_1$ the uniform base generating the standard uniformity (that is, $\mathcal{K}_0$ contains only the diagonal of $\mathfrak{C} \times \mathfrak{C}$, while $\mathcal{K}_1 = \{K_n \colon n = 1, 2, \ldots\}$, where $K_n = \{(x,y) \colon x|n = y|n\}$). Further, for every $t \in [0,1]$ we let $\mathcal{S}^t = \mathbb{P}_k \mathcal{S}^k$, where $\mathcal{S}^k = \mathcal{K}_0$ for $k \neq \operatorname{ord} t$ and $\mathcal{S}^k = \mathcal{K}_1$ for $k = \operatorname{ord} t$. Each $\mathcal{S}^t$ is a uniform base for $X = \mathfrak{C}^{\aleph_0}$ (*typographic caution:* now the elements of $X \times T$ will be denoted by $(\tilde{x}, t)$ and not by (x,t) as in the preceding section.). Finally, let $\mathcal{R} = \{R_n \colon n = 1, 2, \ldots\}$ be any uniform base for the standard uniformity in $T = [0,1]$. The condition (*) of the preceding section is satisfied (select $m = \phi(n,t)$ so that $s \in R_m(t)$ implies $s = t$ or $\operatorname{ord} s > n$), and thus we have a (metrizable) uniformity in $X \times T$.

For reasons that will be apparent after reading 3.1 we will use special neighborhoods in $X \times T$ (see 2.2). For every $t \in [0,1]$ and every $\tilde{x} \in X$ let $I_n(t, \tilde{x})$ be a descending local base of t so that each $I_n(t, \tilde{x})$ is an open interval (a,b), containing t and not contaning any dyadic rational of order $\leq n$, except possibly t, and so that a and b are dyadic rational with $\operatorname{ord} a \neq \operatorname{ord} t \neq \operatorname{ord} b$. (Recall here 2.2a: everything before sect. 6 can be obtained by letting simply $I_n(t, \tilde{x}) = I_n(t) = (a,b)$, where $t \in (a,b)$ and a and b are the dyadic rationals of the smallest possible orders such that $b - a = 2^{-n}$.) For $W^t_n(\tilde{x})$ we shall take an $S^t_m(\tilde{x})$, where $m = m(n)$ increases with n and $m >$ orders of a and b.

The explicit formula for $U_n(\tilde{x}, t) = I_n(t, \tilde{x}) \times W^t_n(\tilde{x})$ is:

$$U_n(\tilde{x}, t) = \{(\tilde{y}, s) : s \in I_n(t, \tilde{x}) \text{ and } \tilde{y}(i) = \tilde{x}(i) \text{ for } i \leq n\} \tag{3.1a}$$

if $n < \operatorname{ord} t$, and

$$U_n(\tilde{x}, t) = \{(\tilde{y}, s) : s \in I_n(t, \tilde{x}) \text{ and } \tilde{y}(i) = \tilde{x}(i) \text{ for } i \leq n+1, \; i \neq \operatorname{ord} t;\; \tilde{y}(i)|n = \tilde{x}(i)|n \text{ for } i = \operatorname{ord} t\} \tag{3.1b}$$

if $n \geq \operatorname{ord} t$.

Now let A_n and B_n be two sequences of subsets of $\mathfrak{C}$ and let $\nu(A_n, B_n) = \{(\widetilde{x}, t) \in X \times T : \widetilde{x}(n) \in A_n \text{ for } n \neq \operatorname{ord} t \text{ and } \widetilde{x}(n) \in B_n \text{ for } n = \operatorname{ord} t\}$. We have

3.1. *In $\nu(A,B)$, $(\tilde{y}, s) \in \operatorname{Bdry} U_n(\tilde{x}, t)$ implies*

(a) *$s \in \{a, b\}$ and $\tilde{y}(m) = \tilde{x}(m)$, for $m \leq n$, $m \neq \operatorname{ord} t$;*

and

(b) $\tilde{y}(m) \in A_m \cap B_m$ *for* $m = \text{ord } s$.[7]
Consequently, if $A_n \cap B_n = \emptyset$ *for every* n, *then* ind $\nu(A_n, B_n) = 0$.

Proof. Part (a). If the conclusion fails, then one of the four cases listed below must hold and we simply verify that in each of these cases we can get an n' so that $U_{n'}(\tilde{y}, s)$ is either contained in or disjoint from $U_n(\tilde{x}, t)$.

Case 1: $s \notin I_n(t, \tilde{x}) \cup \{a, b\}$. Take an n' so that $I_{n'}(s, \tilde{y}) \cap I_n(t, \tilde{x}) = \emptyset$. Then $U_{n'}(\tilde{y}, s) \cap U_n(\tilde{x}, t) = \emptyset$.

Case 2: $s = t$. In this case, $U_n(\tilde{x}, t)$ and $U_n(\tilde{y}, s)$ are detemined by the conjunction of the same equalities; therefore $(\tilde{y}, s) \in U_n(\tilde{x}, t)$ implies $U_n(\tilde{x}, t) = U_n(\tilde{y}, s)$ and $(\tilde{y}, s) \notin U_n(\tilde{x}, t)$ implies $U_n(\tilde{x}, t) \cap U_n(\tilde{y}, s) = \emptyset$, so we can take $n' = n$.

Case 3: $s \in I_n(t, \tilde{x})$, $s \neq t$. Now, ord $s > n$, therefore, either $U_n(\tilde{y}, s) \subset U_n(\tilde{x}, t)$ or $U_n(\tilde{y}, s) \cap U_n(\tilde{x}, t) = \emptyset$, so again $n' = n$.

Case 4: $s \in \{a, b\}$, $\tilde{y}(m) \neq \tilde{x}(m)$, $m \leq n, m \neq \text{ord } t$. Observe that $(\tilde{z}, u) \in U_n(\tilde{x}, t)$ implies $\tilde{z}(m) = \tilde{x}(m)$ for $m \neq \text{ord } t$. Thus, if $m \neq \text{ord } s$, then $n' = n$ will do. If $m = \text{ord } t$, then we take an $n' \geq n$ and such that $\tilde{y}(m)|n' \neq \tilde{x}(m)|n'$.

Part (b). $\tilde{y}(m) \in B_m$ and since ord $t \neq$ ord $s = m$, $\tilde{x}(m) \in A_m$. Since $\tilde{y}(m) = \tilde{x}(m)$, $A_m \cap B_m \neq \emptyset$. ☐

Now for a given subset A of $\mathfrak{C}$ we let $\nu\mu = \nu(A, \mathfrak{C})$ (i.e., $A_n = A$ and $B_n = \mathfrak{C}$), $\nu\mu_0 = \nu(A, \mathfrak{C} \setminus A)$. Summarizing

3.2. $\nu\mu$ *is a metric space;* $\nu\mu_0$ *is a dense subspace of* $\nu\mu$ *with* ind $\nu\mu_0 = 0$,

and

3.3. *If* A *is dense in* $\mathfrak{C}$ *and* $\mathfrak{C} \setminus A$ *is of 2nd category everywhere on* $\mathfrak{C}$, *then* $\dim \nu\mu_0 > 0$.

Proof. Repeat the proof of 7.1 in [M2]. ☐

4. CONDITIONS FOR $\text{IND}_c\, \nu\mu_0 > 0$. PROPERTY (**P**)

We shall formulate here a property of a set $A \subset \mathfrak{C}$ which guarantees $\text{ind}_c\, \nu\mu_0 > 0$.

For $m = 1, 2, \ldots$, let τ_m be the product topology for $\mathfrak{C}^{\aleph_0}$ in which the m-th factor carries the standard topology for $\mathfrak{C}$ and all other factors carry the discrete topology, and let $\mathfrak{C}_m^{\aleph_0}$ be the product $A \times \cdots \times \mathfrak{C} \times A \times \cdots$, where $\mathfrak{C}$ stands only on the m-th place.

Consider the following property of subsets A of $\mathfrak{C}$:

(**P**): *For every system* $F_{n,m}$, $n, m = 1, 2, \ldots$, *of subsets of* $A^{\aleph_0}$ *such that each* $F_{n,m}$ *is* τ_m*-closed in* $\mathfrak{C}_m^{\aleph 0}$, *we have* $\limsup_m \bigcup_n F_{n,m} = \bigcap_k \bigcup_{m>k} \bigcup_n F_{n,m} \neq A^{\aleph_0}$.

4.1. Theorem. *If* A *has* (**P**), *then every completion of* $\nu\mu_0$ *contains a copy of* I *and therefore* $\text{ind}_c\, \nu\mu_0 > 0$.

Proof. Every completion of $\nu\mu_0$ contains a subset homeomorphic to a G_δ-subset G of $\nu\mu$ with $\nu\mu_0 \subset G$; it suffices to show that such a G contains a copy of I. We will let $\nu\mu \setminus G = F_1 \cup F_2 \cup \ldots$, where F_n are closed subsets of $\nu\mu$. Note that if ord $d = m$, then $\{\widetilde{x} \in A^{\aleph_0} : (\widetilde{x}, d) \in F_n\}$ is τ_m-closed; consequently, letting $F_{n,m} = \{\widetilde{x} \in A^{\aleph_0} : (\widetilde{x}, d) \in F_n$ for some d with ord $d = m\}$, we have that $F_{n,m}$ is τ_m-closed. Consequently, there is an $\widetilde{x}_0 \in A^{\aleph_0}$ such that $\widetilde{x}_0 \notin \limsup_m \bigcup_n F_{n,m}$; i.e., there is a k so that $\widetilde{x}_0 \notin \bigcup_{m>k} \bigcup_n F_{n,m}$. In other words, $(\widetilde{x}_0, d) \notin F_n$ for

---

[7] In the above, Bdry U_n and U_n are, of course, relativized to $\nu(A, B)$.

every n and every d with $\operatorname{ord} d > k$. If we take a closed interval I_0 which does not contain dyadic rationals of order $\leq k$, then $(\widetilde{x}_0, t) \in G$ for every $t \in I_0$ (note that $(\widetilde{x}_0, t) \in \nu\mu_0 \subset G$ for every dyadic irrational t). This implies that the function $h(\widetilde{x}, t)$ of 2.3 maps $L_{\widetilde{x}} \cap G$ homeomorphically onto a set containing I_0. Accordingly, we have a copy of I_0 and so of I in G. $\square$

5. Set-theoretic perifery: conditions implying (P)

It is difficult to say what will be the final fate of (Q) or (Q_0).

Property (**P**) is very specialized. It is stated here (despite its complexity) because it is the weakest property known to the author under which (Q_0) can be refuted. We shall now indicate simpler properties that imply (**P**) (strictly speaking, the existence of sets with (**P**)).

Consider the statement

S*($\aleph_0$). *If $X \subset \mathfrak{C}$ is a set of cardinality continuum, then the product $X^{\aleph_0}$ cannot be written as $X^{\aleph_0} = F_1 \cup F_2 \cup \dots$, where*

(a) each F_m is a τ_m-F_σ-subset of $X^{\aleph_0}$

and

(b) each F_m is countable on all lines parallel to the m-th axis of $X^{\aleph_0}$,

and the statement S($\aleph_0$) which is identical to S*($\aleph_0$) except that in (a) we assume that all the factors of $X^{\aleph_0}$ carry the discrete topology. Obviously, S($\aleph_0$) implies S*($\aleph_0$) (S($\aleph_0$) can be applied to any set X, not necessarily subset of $\mathfrak{C}$); furthermore

5.1. S*($\aleph_0$) *implies the existence of sets with* (**P**)*; in fact, under* S*($\aleph_0$)*, every subset A of $\mathfrak{C}$ such that card $A = 2^{\aleph_0}$ and $\mathfrak{C} \setminus A$ is Bernstein* [8] *has the property* (**P**)*.*

Proof. Assume that $F_{n,m}$ is a system of subsets of $A^{\aleph_0}$ referred to in (**P**). Let L be a line parallel to the m-th axis and let $F' = \lambda^{-1}(F_{n,m} \cap L)$, where λ is the canonical map of $\mathfrak{C}$ onto L. Since $F_{n,m}$ is closed in $\mathfrak{C}_m^{\aleph 0}$, F' is a subset of A which is closed in $\mathfrak{C}$, and since $\mathfrak{C} \setminus A$ is Bernstein, F' and therefore also $F_{n,m} \cap L$ is countable. Thus $F_m = \bigcup_n F_{n,m}$ is an F_σ-set and it is countable on L. By S*($\aleph_0$) we cannot have $\bigcup_m F_m = A^{\aleph_0}$ and *a fortiori* we cannot have $\limsup_m \bigcup_n F_{n,m} = A^{\aleph_0}$. $\square$

S($\aleph_0$) is a very simple set-theoretic statement. It is obviously related to the classical result of Sierpiński S(m): *The product X^m can be written as $X^m = A_1 \cup \dots \cup A_m$, where each A_i is of cardinality $< \aleph_\alpha$ on all lines parallel to the i-th axis iff* card $X < \aleph_{\alpha+m}$ (m is finite; S(m) does not involve any topology). S($\aleph_0$), as well as the problem of its consistency, has been stated in the list of problems to [M2]; S*($\aleph_0$) and the property (**P**) are stated here for the first time. R. Dougherty has devoted his doctoral dissertation to this subject; unforunately, he has obtained consistency of some modifications of S($\aleph_0$) (of course, one speaks here about very relative consistency – thus, more precisely, one should say that Dougherty was able to *place* some modifications of S($\aleph_0$) among conditions discussed in the literature). He also points out that S($\aleph_0$) requires $2^{\aleph_0}$ to be Banach measurable. But S*($\aleph_0$) and *a fortiori* the existence of sets with the property (**P**) could be consistent under less drastic situations. In any case it seems that the Set Theory as well as the theory of Borel sets have gained interesting problems.

---

[8] I.e., $\mathfrak{C} \setminus A$ intersects every perfect subset of $\mathfrak{C}$ – in simpler terms: every compact subset of A is countable.

If the existence of sets with (**P**) is false, one could still attempt to refute (Q_0) by switching to higher cardinalities. For brevity, consider the condition $\mathrm{C}(\mathfrak{m}, \mathfrak{n}, \mathcal{K}_n)$, where $\mathfrak{m}$ and $\mathfrak{n}$ are cardinals and $\mathcal{K}_n$ are properties of sets:
$\mathrm{C}(\mathfrak{m}, \mathfrak{n}, \mathcal{K}_n)$: *If X is a set of cardinality $\mathfrak{m}$, then $X^{\aleph_0}$ cannot be written as $X = F_1 \cup F_2 \cup \dots$, where each F_n has $\mathcal{K}_n$ and is of cardinality $< \mathfrak{n}$ on all lines parallel to the n-th axis.*

Thus $\mathrm{S}(\aleph_0)$ is $\mathrm{C}(2^{\aleph_0}, \aleph_1, \tau\text{-}F_\sigma)$ (τ – the product topology for $X^{\aleph_0}$ with all factors being discrete) and $\mathrm{S}^*(\aleph_0)$ – which is applicable only to subsets of the Cantor set – is $\mathrm{C}(2^{\aleph_0}, \aleph_1, \tau_n\text{-}F_\sigma)$.

Now let $\mathfrak{m}_0$ be a strong limit cardinal of countable cofinality and let $B_{\mathfrak{m}_0}$ be the 0-dimensional Baire space of weight $\mathfrak{m}_0$ (i.e., the product of countably many discrete spaces of cardinality $\mathfrak{m}_0$). Let $\mathrm{S}(\mathfrak{m}_0)$ and $\mathrm{S}^*(\mathfrak{m}_0)$ stand for $\mathrm{C}(2^{\mathfrak{m}_0}, \mathfrak{m}^+, \tau\text{-}F_\sigma)$ and $\mathrm{C}(2^{\mathfrak{m}_0}, \mathfrak{m}^+, \tau_n\text{-}F_\sigma)$, respectively; $\mathrm{S}^*(\mathfrak{m}_0)$ is applicable only to subsets of $B_{\mathfrak{m}_0}$. The construction of $\nu\mu_0$ can be reproduced in $B_{\mathfrak{m}_0}$ (in place of the Cantor set) and in this way we obtain a space for which c-spread > 0 is implied by $\mathrm{S}^*(\mathfrak{m}_0)$.

6. $\operatorname{ind}_c \nu\mu_0 = 0$ UNDER (CH)

Here we show the other side of the coin[9] : $\operatorname{ind}_c \nu\mu_0 > 0$ certainly is *not* a theorem of ZFC.

6.1. *The summation $\sum$ of uniform bases preserves completeness.*

Proof. Observe the formulas

$$(\tilde{x}, t) P_n (\tilde{y}, s) \quad \text{implies} \quad t R_n s, \tag{6.1}$$

and

$$(x, t) P_{\psi(n,u)} (y, s) \quad \text{and} \quad t\, R_{\psi(n,u)+1} u \quad \text{imply} \quad x\, S^u_n y. \tag{6.2}$$

(6.1) is obvious. To verify (6.2) observe that there is an (r, v) so that $t R_{\psi(\psi(n,u),v)} v$ and $x S^v_{\psi(\psi(n,u),v)} r$, $r S^v_{\psi(\psi(n,u),v)} y$. Since $\psi(\psi(n,u),v) > \psi(n,u)$, $t R_{\psi(n,u)+1} v$ and therefore

$$v R_{\psi(n,u)} u. \tag{6.3}$$

Furthermore, $x\, S^v_{\psi(\psi(n,u),v)-1} y$; therefore $x\, S^v_{\psi(n,u)} y$, and so, by (6.3) and (*), $S^v_{\psi(n,u)} \subset S^u_n$, thus $x S^u_n y$ (we make use of the obvious fact $\psi(n,u) > \phi(n,u)$) and (6.2) is shown.

Now, let (x_n, t_n) be a $\mathcal{P}$-Cauchy sequence. (6.1) implies that t_n is $\mathcal{R}$-Cauchy; let $t_n \to u$. (6.2) shows that x_n is $\mathcal{S}^u$-Cauchy and the rest is obvious. $\square$

By the above, $X \times T$ is complete. $\nu\mu$ is closed in $X \times T$ and therefore $\nu\mu$ is a completion of $\nu\mu_0$. Consequently, it suffices to find a G_δ-subset K of $\nu\mu$ containing $\nu\mu_0$ and such that $\operatorname{ind} K = 0$. To this end we will use the Sierpiński decomposition of $A \times A$ formulated as follows: there exists a function $f : N \times A \to A$ such that

$$\text{for every } a, b \in A \text{ there is a } k \text{ so that} \quad a = f(2k, b) \text{ or } b = f(2k+1, a). \tag{6.4}$$

---

[9] It appears that the procedure presented in this section allows one to obtain $\operatorname{ind}_c \nu\mu_0 = 0$ under $2^{\aleph_0} < \aleph_\omega$. However, I have not yet verified the details.

Define for $n, k = 1, 2, \ldots,\ i = 0, 1$,

$$(6.5)\quad F_{n,k,i} = \{(\tilde{x}, d) :\ \tilde{x}(2n+i) = f(2k+i, \tilde{x}(2n-i+1)) \text{ and } \operatorname{ord} d = 2n+i\}.$$

6.2. $F_{n,k,i}$ *is closed.*

Let $(\tilde{x}_0, t_0) \notin F_{n,k,i}$. If $\operatorname{ord} t_0 \neq 2n+i$, then take an m so that $t \notin I_m(\tilde{x}_0, t_0)$ for every t with $\operatorname{ord} t = 2n+i$. Then $U_m(\tilde{x}_0, t_0) \cap F_{n,k,i} = \emptyset$. If $\operatorname{ord} t_0 = 2n+i$, then we must have $\tilde{x}_0(2n+i) \neq f(2k+i, \tilde{x}_0(2n-i+1))$. Take an $m > 2n+1$ so that $\tilde{x}_0(2n+i)|m \neq f(2k+i, \tilde{x}_0(2n-i+1))|m$. Then $(\tilde{x}, t) \in U_m(\tilde{x}_0, t_0)$ implies $\tilde{x}(2n-i+1) = \tilde{x}_0(2n-i+1)$ and $\tilde{x}(2n+i)|m = \tilde{x}_0(2n+i)|m$; therefore $\tilde{x}(2n+i)|m \neq f(2k+i, \tilde{x}(2n-i+1))|m$ and hence $(\tilde{x}, t) \notin F_{n,k,i}$. Thus $U_m(\tilde{x}_0, t_0) \cap F_{(n,k,i)} = \emptyset$, so $F_{(n,k,i)}$ is closed.

Let $K = \nu\mu \setminus \bigcup_{n,k,i} F_{n,k,i}$. Since $F_{n,k,i}$ are disjoint from $\nu\mu_0$, K is a G_δ containing $\nu\mu_0$. It remains to show that $\operatorname{ind} K = 0$. Now $U_n(\tilde{x}, t)$ have to be selected with greater care. Given an $(\tilde{x}, t) \in K$, we have, by (6.4), that for every n there are k_n and i_n so that $\tilde{x}(2n+i_n) = f(2k_n+i_n, \tilde{x}(2n-i_n+1))$. Let $I_n(t, \tilde{x})$ be the shortest open interval containing t and so that the endpoints of $I_n(t, \tilde{x})$ are dyadic rationals of order $2n+i_n$; if t is a dyadic rational, then we do this construction only for $n > \operatorname{ord} t$. (Recall 2.2a: here the $I_n(t, \tilde{x})$'s depend upon $\tilde{x}$; in fact, in a quite intricate way. The very definition of $I_n(t, \tilde{x})$ involves – through the Sierpiński decomposition – the continuum hypothesis.) If a point $(\tilde{y}, s)$ of K belongs to $\operatorname{Bdry} U_n(\tilde{x}, t)$, then, by 3.1, s is one of the endpoints of $I_n(\tilde{x}, t)$, thus s is a dyadic rational of order $2n+i_n$; furthermore, $\tilde{y}(2n+i) = \tilde{x}(2n+i)$ for $i = 0, 1$, therefore $\tilde{y}(2n+i_n) = f(2k_n+i_n, \tilde{y}(2n-i_n+1))$, thus $(\tilde{y}, s) \in F_{n,k_n,i_n}$. This contradiction shows that $\operatorname{Bdry} U_n(\tilde{x}, t) = \emptyset$. (The $I_n(\tilde{x}, t)$'s, as defined above, need not be descending, thus one has to pass to subsequence. But this concludes the proof that $\operatorname{ind} K = 0$.)

7. Problems

It is known that the condition d-spread > 0 can be realized within complete spaces (through Δ) as well as within N-compact spaces (through μ_0).

Problem 1. *Does there exist a complete N-compact space with* d-spread > 0*? In particular, decide whether μ_0 has an N-compact completion.*

It is known that within topological (completely regular) spaces with $\operatorname{ind} = 0$ the condition

(coB): *every functionally closed set is co-Borel*

(where co-Borel stands for the σ-algebra generated by clopen sets) is strictly weaker than $\dim = 0$. It is also known that (R-compact) + (coB) $\Longrightarrow$ (N-compact).

Problem 2. *In metric spaces,*

(a) *does* (N-compact) *imply* (coB)*?*

(b) *does* (N-compact) + (coB) *imply* $\dim = 0$*?*

(NO *to both in non-metric spaces*).

In particular, verify (coB) *in μ_0 and μ_{00}.*

8. Added after submission of the paper

Sent to the Editor on March 14, 1996:

As it was emphasized in sect. 1, publishing results based on statements of unproven consistency involves an obvious risk. The author is pleased to report that in this case the risk was worth taking – R. Dougherty at this time was able to prove the (relative) consistency of $\mathrm{S}(\aleph_0)$ (e-mail message received on January 23, 1996). Thus the (relative) consistency of the negation of (Q_0) is no longer a matter of speculation. In fact, $\mathrm{S}(\aleph_0)$ implies other significant properties of $\nu\mu_0$ and in a few months we should be able to publish an account of these considerations – consistency of answers to other basic problems in the dimension theory of metric spaces.

It is still unclear how far we are from the final settlement of these problems. The step from consistency to validity might be an enormous one. In fact, it is possible – this, at the present moment, is a pure speculation – that these problems cannot be answered under the usual axioms of Set Theory. But even if they can – e.g., if there exists a space for which, in ZFC, $\operatorname{ind} = 0$ and $\operatorname{ind}_c = 1$, the permanent effect of these considerations is the existence of spaces for which ind_c cannot be determined in ZFC. (But note that, in ZFC, $\dim \nu\mu_0 = 1$; on the other hand, it appears that neither $\operatorname{ind}_c \nu\mu_0^2$ *nor* $\dim \nu\mu_0^2$ can be determined in ZFC.) More detailed comments on the emerging situation will be given in a sequel to this paper.

(Dougherty notes that my quote *“He (Dougherty) also points out that $S(\aleph_0)$ requires $2^{\aleph_0}$ to be Banach measurable”* was incorrect – $\mathrm{S}(\aleph_0)$ requires somewhat less.)

References

[Do] R. Dougherty, *Narrow coverings of ω-product spaces*, Ph.D. Dissertation, U. of C., Berkeley, 1984.

[M1] S. Mrówka, *Recent results on E-compact spaces*, Proc. 2nd Int. Conf., General Topology and Its Applications, 1972. MR **50:**14673

[M2] ———, *N-compactness, metrizability and covering dimension*, Rings of continuous functions, Marcell Dekker, Inc., New York and Basel, 1985, pp. 248 - 275. MR **86i:**54034

[Pe] A.R. Pears, *Dimension Theory of General Spaces*, Cambridge, 1975. MR **52:**15405

[R1] P. Roy, *Failure of equivalence of dimension concepts for metric spaces*, Bull. A.M.S. **68** (1962), 609 - 613. MR **25:**5495

[R2] ———, *Non-equality of dimensions for metric spaces*, Trans. A.M.S. **134** (1968), 117 - 132. MR **37:**3544

Department of Mathematics, SUNY at Buffalo, 134 Defendorf Hall, Buffalo, New York 14224

E-mail address: `mrowka@acsu.buffalo.edu`