ALL MAPS OF TYPE $2^\infty$ ARE BOUNDARY MAPS

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Abstract. Let $f$ be a continuous map of an interval into itself having periodic points of period $2^n$ for all $n \geq 0$ and no other periods. It is shown that every neighborhood of $f$ contains a map $g$ such that the set of periods of the periodic points of $g$ is finite. This answers a question posed by L. S. Block and W. A. Coppel.

1. Introduction

Let $I$ be a real compact interval and $C(I, I)$ be the metric space of continuous maps of $I$ into itself with the distance between two elements $f, g$ defined by $g(f, g) = \max\{|f(x) - g(x)| : x \in I\}$. Let $\mathbb{N}$ be the set of positive integers. A point $p \in I$ is a periodic point of a map $f$ if $f^n(p) = p$ for some $n \in \mathbb{N}$. The period of $p$ is the least such integer $n$, and the orbit of $p$ under $f$ is the set $\{f^k(p) : k = 0, 1, \ldots, n - 1\}$. We refer to such an orbit as a periodic orbit of $f$ of period $n$.

A map $f \in C(I, I)$ is piecewise monotone if there are points $\min I = a_0 < a_1 < \cdots < a_n = \max I$ such that for every $k \in \{1, 2, \ldots, n\}$, the restriction of $f$ to the interval $[a_{k-1}, a_k]$ is (not necessarily strictly) monotone. When speaking of a piecewise monotone map we can always take $n$ as the minimal positive integer with this property and call the points $a_1, \ldots, a_{n-1}$ turning points of $f$ (though they still are not uniquely determined by $f$).

Consider the Sharkovskii ordering of the set $\mathbb{N} \cup \{2^\infty\}$:

$$3 \succ 5 \succ 7 \succ \cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \cdots \succ 4 \cdot 3 \succ 4 \cdot 5 \succ 4 \cdot 7 \succ \cdots \succ \cdots$$

$$\succ 2^\infty \cdot 3 \succ 2^\infty \cdot 5 \succ 2^\infty \cdot 7 \succ \cdots \succ 2^\infty \succ \cdots \succ 2^\infty \succ \cdots \succ 2^\infty \succ 4 \succ 2 \succ 1.$$

We will also use the symbol $\succ$ in the natural way. For $t \in \mathbb{N} \cup \{2^\infty\}$ we denote by $S(t)$ the set \{\{\{k \in \mathbb{N} : t \geq k\}\} \{\{2^\infty\}\}$ stands for the set \{1, 2, 4, 2^k, \ldots\}). Let $f \in C(I, I)$ and $\text{Per}(f)$ be the set of periods of its periodic points.

Sharkovskii’s Theorem ([Sh1],[Sh2]). For every $f \in C(I, I)$ there exists a $t \in \mathbb{N} \cup \{2^\infty\}$ with $\text{Per}(f) = S(t)$. On the other hand, for every $t \in \mathbb{N} \cup \{2^\infty\}$ there exists an $f \in C(I, I)$ with $\text{Per}(f) = S(t)$.

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1667
If $\text{Per}(f) = S(t)$, then $f$ is said to be of type $t$. When speaking of types we consider them to be ordered by the Sharkovskii ordering. So if a map $f$ is of type $2^\infty$ or greater than $2^\infty$ or less than $2^\infty$, then, respectively, $\text{Per}(f) = \{1, 2, \ldots, 2^n, \ldots\}$ or $f$ has a periodic point with period not a power of 2 or $\text{Per}(f) = \{1, 2, \ldots, 2^n\}$ for some $N$. The set $\text{Per}(f)$ is finite if and only if $f$ is of type less than $2^\infty$. The topological entropy of $f$ is positive if and only if $f$ is of type greater than $2^\infty$ (see [BF] for the “if” part and [Mi] for the “only if” part or [ALM, Theorem 4.4.19]).

Recall that it is very easy to see that any neighborhood of any map $f$ contains maps of types greater than $2^\infty$ (even maps of type 3) (see [Kl]). Contrary to the maps of types greater than $2^\infty$, the maps of types less than $2^\infty$ do not form a dense set in $C(I, I)$. In fact, this set is nowhere dense in $C(I, I)$. To see this use the following Block’s Theorem.

**Block’s Theorem** ([Bl]). Let $f \in C(I, I)$ and let $n \in \text{Per}(f)$. Then there exists a neighborhood $U(f, n)$ of $f$ such that for all $g \in U(f, n)$ we have $\text{Per}(g) \supset S(n) \setminus \{n\}$.

So, if $f$ is of type greater than $2^\infty$, then there is a neighborhood of $f$ containing no map of type less than (or equal to) $2^\infty$. L. S. Block and W. A. Coppel (see [BC], the end of chapter II.4) posed the question of whether any neighborhood of any map of type $2^\infty$ contains a map of type less than $2^\infty$. We answer this question in the affirmative by proving the following

**Theorem.** Let $f \in C(I, I)$ be of type $2^\infty$. Then every neighborhood of the map $f$ contains a piecewise monotone map of type less than $2^\infty$.

We prove this theorem in two steps. First we prove it under the additional assumption that $f$ is piecewise monotone. Then we prove that in any neighborhood of a map of type $2^\infty$ there is a piecewise monotone map of type at most $2^\infty$. But before going to the proof we wish to mention some aspects of this theorem.

Denote by $G$ or $E$ or $L$, respectively, the set of all maps of types greater than or equal to or less than $2^\infty$. For any set $A$ in the metric space $C(I, I)$ let $\text{Bd} A$ denote the boundary of $A$. Our theorem and Block’s Theorem show that $\text{Bd} G = \text{Bd} L = E \cup L$. This is what is meant by the title of this paper.

In the Sharkovskii ordering the smallest element is 1 and the largest one is 3. For any $n \in \mathbb{N} \setminus \{1\}$ denote by $\nu(n)$ the predecessor of $n$, i.e., the maximum (in the Sharkovskii ordering) of the set $S(n) \setminus \{n\}$. If $f$ is of type $n \in \mathbb{N} \setminus \{1\}$ then, by Block’s Theorem, there is a neighborhood $U$ of $f$ such that for every $g \in U$, the type of $g$ is at least $\nu(n)$. It is also known (and not difficult to show) that for any $n \in \mathbb{N}$ there exist a map $f_n$ of type $n$ and a neighborhood $U$ of $f_n$ such that for every $g \in U$, the type of $g$ is at least $n$. Our theorem shows that this is not true for $n = 2^\infty$. In other words, if $f$ is of type $2^\infty$, and if for every $n \in \mathbb{N}$ the neighborhood $U_n$ of $f$ satisfies that every map in $U_n$ is at least of type $2^n$ (the existence of such neighborhoods follows from Block’s Theorem), then $\bigcap_{n=0}^{\infty} U_n = \{f\}$.

The Sharkovskii Theorem holds also for the set $C(\mathbb{R}, \mathbb{R})$ of continuous maps from the real line $\mathbb{R}$ into itself (in this case we have the additional possibility $\text{Per}(f) = \emptyset$) (see, e.g., [ALM, Corollary 2.1.2]). It is known that Block’s Theorem works also for maps from $C(\mathbb{R}, \mathbb{R})$ (now $g(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$ may be infinite) (see [ALM], Remark 2.8.5). So, it is natural to ask whether at least the weaker form of the theorem without the words “piecewise monotone” is true in $C(\mathbb{R}, \mathbb{R})$. The answer is negative. In fact, for any $n \in \mathbb{N}$ take a map $\varphi_n \in C([0, 1], [0, 1])$ of type $2^n$ with $\varphi_n([0, 1]) \subset [1/4, 3/4]$ and $0 < \varepsilon_n < 1/4$ such that the ball $B(\varphi_n, \varepsilon_n)$
contains only maps with types at least $2^{n-1}$. Then there are $a_n > 0$ and a map $\alpha_n \in C([0, a_n], [0, a_n])$ of type $2^n$ with $\alpha_n ([0, a_n]) \subset [1, a_n - 1]$ such that the ball $B(\alpha_n, 1)$ contains only maps with types at least $2^{n-1}$. Take $a_n = 1/\epsilon_n$ and $\alpha_n = h_1 \circ \varphi_n \circ h_n^{-1}$ where $h_n$ is the increasing affine map of $[0, 1]$ onto $[0, a_n]$. Further, put $b_0 = 0, b_1 = 1$ and $b_{2n} = n + \sum_{i=1}^{n} a_i, b_{2n+1} = b_{2n+1}$ for $n = 1, 2, \ldots$ and let $\beta_n \in C([b_{2n-1}, b_{2n}], [b_{2n-1}, b_{2n}])$ be defined by $\beta_n = k_n \circ \alpha_n \circ k_n^{-1}$ where $k_n$ is the increasing affine map from $[0, a_n]$ onto $[b_{2n-1}, b_{2n}]$. Then $f \in C(\mathbb{R}, \mathbb{R})$ defined by

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ \beta_n(x), & \text{if } x \in [b_{2n-1}, b_{2n}], \\ \text{affine,} & \text{if } x \in [b_{2n-2}, b_{2n-1}] \end{cases}$$

is a map from $C(\mathbb{R}, \mathbb{R})$ of type $2^n$ such that all maps from the ball $B(f, 1)$ have types at least $2^n$ (use that $g([b_{2n-1}, b_{2n}]) \subset [b_{2n-1}, b_{2n}]$ whenever $g \in B(f, 1)$).

2. PROOF OF THE THEOREM

If $A \subset I$, then int $A$ or diam $A$ will denote the interior or the diameter of $A$, respectively. If $J_1, J_2 \subset I$ are intervals, then $J_1 < J_2$ will mean that $\sup J_1 < \inf J_2$. If $f$ is a map and $A$ is a set, then $f|_A$ is the restriction of $f$ to $A$. Let $f \in C(I, I)$. We say that $x \in I$ is eventually periodic if $f^n(x)$ is periodic for some $n$.

Given $f \in C(I, I)$, a closed subinterval $J$ of $I$ is periodic of period $n$ if $f^n(J) = J$ and $f^k(J) \cap f^{k+1}(J) = \emptyset$ for any $0 \leq k < l < n$. Further, we say that $S \subset I$ is a (simple) solenoid of $f$ if $S = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} f^k(I^n)$ where for any $n, I^n$ is a periodic interval of period $2^n$ such that $I^n \supset I^{n+1}$. The equality $S = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} f^k(I^n)$ will be said to be a standard representation of the solenoid $S$. Clearly, $S$ is a compact set, $f(S) = S$ and $S$ cannot contain any eventually periodic point of $f$.

Lemma 1. Let $f \in C(I, I)$ and let $R, S$ be different solenoids of $f$. Then $R \cap S = \emptyset$.

Proof. This follows immediately from the proofs of Proposition 2.2 (7) and Lemma 2.15 in [Pr] (these proofs work without the assumption of piecewise monotonicity).

Lemma 2. Let $f \in C(I, I)$ be piecewise monotone and $S$ be a solenoid of $f$. Then $S$ must contain a turning point. In particular the number of solenoids of $f$ is finite.

Proof. If $S = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} f^k(I^n)$ is a standard representation of a solenoid $S$ of $f$, then, taking any $n$, $f$ cannot be monotone on all intervals $f^k(I^n), k = 0, 1, \ldots, 2^n - 1$. Consequently, there is at least one of the finitely many turning points of $f$ belonging to $S$. The rest of the lemma follows from Lemma 1.

The set of all limit points of the trajectory $(f^n(x))_{n=0}^{\infty}$ of a point $x$ is called the $\omega$-limit set of $x$ under $f$ and is denoted by $\omega_f(x)$. A standard well-known result says that every finite $\omega$-limit set is a periodic orbit.

A well-known result implicit in several of Sharkovskii’s papers and proved in [Sm] (see also [FS]) states that every infinite $\omega$-limit set of a map of type $2^{\infty}$ is contained in a solenoid. In [Ge] it is proved that if a map $f$ of type $2^{\infty}$ is additionally piecewise monotone, then any infinite $\omega$-limit set of $f$ is Cantor-like. Finally, we will substantially use a result from [JS] saying that every piecewise monotone map of type $2^{\infty}$ must have an infinite (Cantor-like) $\omega$-limit set. Combining these facts we get
Lemma 3. Every piecewise monotone map $f \in C(I,I)$ of type $2^\infty$ has infinite $\omega$-limit sets, each of them being Cantor-like and contained in a solenoid.

Now we are able to prove our main result under the additional assumption that $f$ is piecewise monotone.

Lemma 4. Let $f \in C(I,I)$ be of type $2^\infty$ and piecewise monotone. Then every neighborhood of $f$ contains a piecewise monotone map of type less than $2^\infty$.

Proof. Let $\varepsilon > 0$. We wish to find a map $g \in C(I,I)$ of type less than $2^\infty$ with $g(f,g) < \varepsilon$. Take $\delta > 0$ such that $\text{diam } f(J) < \varepsilon$ whenever $\text{diam } J < \delta$.

Special case. Assume that the map $f$ has only one solenoid $S$ (see Lemma 3 and Lemma 2).

In the standard representation $S = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^{n-1}} I^*_k$, $I^*_k = f^k(I^n)$, we may assume that $I^*_0 \supset I^*_0^{n+1}$ for every $n$. Then $I^*_{k+1}^{n+1}, I^*_{k+2^{n}} \subset I^*_k$ for every $n$ and $k = 0, 1, \ldots, 2^n - 1$. Denote by $K^*_k$ the (maximal) closed interval lying between $\text{int } I^*_{k+1}$ and $\text{int } I^*_{k+2^{n}}$. Further, for every $n$ denote $I(n) = \{ I^*_k : k = 0, 1, \ldots, 2^n - 1 \}$, $K(n) = \{ K^*_k : k = 0, 1, \ldots, 2^n - 1 \}$ and $\bigcup I(n) = \bigcup_{k=0}^{2^n-1} I^*_k$, $\bigcup K(n) = \bigcup_{k=0}^{2^n-1} K^*_k$.

Realize the following:

(i) There is a (sufficiently large) $l_1$ such that $\bigcup I(l_1)$ contains only those turning points of $f$ which belong to $S$. Then, for any $n \geq l_1$, $\text{int } (\bigcup K(n))$ does not contain any turning point of $f$.

(ii) There is an $l_2$ such that $I(l_2)$ contains an interval with diameter less than $\delta$.

Take $l = \max \{ l_1, l_2 \}$. Then $I(l)$ contains an interval, say $I^*_0$, with diameter less than $\delta$. Further, $f$ is monotone on each of the intervals belonging to $K(l)$.

We have $P = I^*_0 + 1 \cup K^*_0 \cup I^*_2 + 1 \subset I^*_0$ where we may assume that $I^*_0 + 1 < I^*_2 + 1$.

Define $g \in C(I,I)$ by

$$g(x) = \begin{cases} f(x), & \text{for } x \in I \setminus (K^*_0 \cup I^*_2 + 1), \\ f(\max I^*_2 + 1), & \text{for } x \in I^*_2 + 1, \\ \text{affine on } K^*_0. \end{cases}$$

The map $g$ is piecewise monotone and since $\text{diam } f(P) < \varepsilon$, $g(P) \subset f(P)$ and $f$ coincides with $g$ on $I \setminus P$, we have $g(f,g) < \varepsilon$. To finish the proof of the Special case, it is sufficient to show that $g$ is of type less than $2^\infty$.

Take any $x \in I$. If the trajectory $(g^n(x))_{n=0}^\infty$ does not visit $K^*_0 \cup I^*_2 + 1$, then it coincides with the trajectory $(f^n(x))_{n=0}^\infty$, whence $\omega_g(x) = \omega_f(x)$. Since the set $\omega_g(x) = \omega_f(x)$ is not a subset of $S$ (otherwise the trajectory $(g^n(x))_{n=0}^\infty$ would intersect $I^*_2 + 1$), it is a finite set, a periodic orbit of $f$ (as well as of $g$) of period a power of 2.

Now suppose that $(g^n(x))_{n=0}^\infty$ visits the set $I^*_2 + 1$. Then the point $x$ is eventually periodic with period $2^l + 1$ and so $\omega_g(x)$ is finite.

Finally suppose that $(g^n(x))_{n=0}^\infty$ does not visit $I^*_2 + 1$ and, for some $n_0$, $g^{n_0}(x) \in K^*_0$. Then $(g^n(x))_{n=n_0}^\infty$ lies in $\bigcup K(l)$. Due to the fact that $g$ is monotone on each of the intervals belonging to $K(l)$, for any $k \in \{ 0, 1, \ldots, 2^n - 1 \}$ the set $M^*_k = \{ y \in K^*_k : (g^n(y))_{n=0}^\infty \text{ lies in } \bigcup K(l) \}$ is an interval (possibly degenerate), the map $h = g^2$ maps this interval into itself and is monotone on it. Hence, for any $y \in M^*_k$ the cardinality of $\omega_h(y)$ is at most 2 (see [Co] or [Sh2]). Therefore, for our point $x$ we get that $\omega_g(x)$ has cardinality $2^l$ or $2^{l+1}$.
We have proved that for every \( x \in I, \omega_g(x) \) is finite and, moreover, has cardinality a power of 2. So \( g \) is of type at most \( 2^\infty \). It cannot be of type \( 2^\infty \) since otherwise, being piecewise monotone, by [JS] (cf. Lemma 3) it would have an infinite \( \omega \)-limit set. Therefore \( g \) is of type less than \( 2^\infty \).

**General case.** The map \( f \) has the solenoids \( S_1, S_2, \ldots, S_r \) (and no other ones) for some \( r \in \mathbb{N} \).

Since the solenoids \( S_i, i = 1, 2, \ldots, r \), are compact and mutually disjoint, each of them has a positive distance from the others. So, if \( S_i = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{\infty} (I_k)^n_i \) is a standard representation of \( S_i \), there exists an \( m \) such that the sets \( \bigcup (I_k)(m) = \bigcup_{k=0}^{\infty} (I_k)^m_i, i = 1, 2, \ldots, r \), are pairwise disjoint. Now the same procedure which was applied in the Special case to a map having one maximal solenoid \( S \) can be applied \( r \)-times to our map \( f \) having \( r \) solenoids \( S_1, S_2, \ldots, S_r \). As a result we get a piecewise monotone map \( g \) of type less than \( 2^\infty \) with \( g(f, g) < \varepsilon \).

In what follows, \( h(f) \) denotes the topological entropy of \( f \) (see [AKM] or [ALM] for the definition and properties). Here we just recall that \( h(f) \in [0, +\infty] \).

**Lemma 5.** Let \( X \) be a compact metric space, and let \( T : X \to X \) be a continuous map. Suppose that \( (U_n)_{n \geq 1} \) is a sequence of open subsets of \( X \), and let \( S : X \to X \) be a continuous map such that \( S(x) = T(x) \) for all \( x \in X \setminus \bigcup_{n=1}^{\infty} U_n \) and \( S|_{U_n} \) is constant for all \( n \geq 1 \). Then \( h(S) \leq h(T) \).

**Proof.** Set \( V := \bigcup_{k=0}^{\infty} \bigcup_{n=1}^{\infty} S^{-k}(U_n) \) and \( C := X \setminus V \). Obviously \( S(C) \subseteq C \) and \( S|_C = T|_C \). An easy calculation shows that \( \Omega(S) \cap V \) equals an at most countable union of periodic orbits \( (\Omega(S) \cap V \subseteq \emptyset) \), where \( \Omega(S) \) denotes the nonwandering set of \( S \) (see [W] for the definition and properties). The variational principle (see [W, Theorem 8.6 and Corollary 8.6.1] and [W, Theorem 6.15]) give

\[
h(S) = \begin{cases} h(S|_C), & \text{if } C \neq \emptyset, \\ 0, & \text{otherwise}. \end{cases}
\]

In the first case \( h(S|_C) = h(T|_C) \leq h(T) \), and in the second case \( h(S) \leq h(T) \) is trivial.\( \square \)

**Remark.** An easy consequence of Lemma 5 is the following fact. If \( f, g \in C(I, I) \) coincide outside an open set \( G \), and if \( g \) is constant on every connected component of \( G \), then \( h(g) \leq h(f) \).

**Lemma 6.** Let \( f \in C(I, I) \) and let \( \{u, v\} \subseteq I \) be a closed interval with max \( f([u, v]) \in \{f(u), f(v)\} \). Then there is a map \( g \in C(I, I) \) with the following properties:

1. \( g(x) = f(x) \) whenever \( x \notin (u, v) \),
2. \( g|_{[u,v]} \) is (not necessarily strictly) monotone, and
3. \( h(g) \leq h(f) \).

**Proof.** Assume that \( \max f([u, v]) = f(v) \) (if the maximum is attained at the point \( u \), the proof is analogous). Define the map \( g|_{[u,v]} \) as the so-called rising sun function corresponding to the map \( f|_{[u,v]} \) when a rising sun is on the \( x \)-axis at \(-\infty \) (equivalently, pour water into the graph of \( f|_{[u,v]} \) until you get all “holes in the ground” full of water). More precisely, put

\[
g(x) = \begin{cases} f(x), & \text{if } x \in I \setminus [u, v], \\ \max \{f(t) : u \leq t \leq x\}, & \text{if } x \in [u, v]. \end{cases}
\]
Then (1) and (2) hold trivially, the continuity of $g$ follows from the assumption that $\max f([u,v]) = f(v)$. Finally, the set $G = \{x \in [u,v] : f(x) < g(x)\}$ is open, $g$ equals $f$ on $I \setminus G$ and $g$ is constant on every connected component of $G$. So, Lemma 5 gives (3).

\begin{lemma}
Let $f \in C(I,I)$. Then in every neighborhood of $f$ there is a piecewise monotone map $g$ with $h(g) \leq h(f)$.
\end{lemma}

\begin{proof}
Let $\varepsilon > 0$. Take $\delta > 0$ such that $\text{diam } f(J) < \varepsilon$ whenever $\text{diam } J < \delta$. Now take points $\min I = z_1 < z_2 < \cdots < z_k = \max I$ with $|z_i - z_{i+1}| < \delta$ for $i = 1, 2, \ldots, k - 1$. In each interval $[z_i, z_{i+1}]$ take a point $s_i$ such that $f(s_i) = \max f([z_i, z_{i+1}])$. Let

$\{z_i : i = 1, 2, \ldots, k\} \cup \{s_i : i = 1, 2, \ldots, k - 1\} = \{x_1, x_2, \ldots, x_n\}$

with $\min I = x_1 < x_2 < \cdots < x_n = \max I$. Since each of the intervals $[x_i, x_{i+1}]$, $i = 1, 2, \ldots, n - 1$, can be viewed as the interval $[u, v]$ in Lemma 6, we can use the lemma $n - 1$ times to get a piecewise monotone map $g$ with $h(g) \leq h(f)$ and $g(f,g) < \varepsilon$.

\end{proof}

\begin{proof}[Proof of Theorem]
Since for $\varphi \in C(I,I)$ we have $h(\varphi) = 0$ if and only if $\varphi$ is of type at most $2^\infty$, it suffices to use Lemma 7 and Lemma 4.
\end{proof}

\begin{references}


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