NONSTANDARD MODELS
AND ANALYTIC EQUIVALENCE RELATIONS

SY D. FRIEDMAN AND BOBAN VELICKOVIC

(Communicated by Andreas R. Blass)

Abstract. We improve a result of Hjorth concerning the nature of thin analytic equivalence relations. The key lemma uses a weakly compact cardinal to construct certain nonstandard models, which Hjorth obtained using #'s.

The purpose of this note is to improve the following result of Hjorth [2].

Theorem (Hjorth). Suppose that for every real \( x \), \( x^\# \) exists. Let \( E \) be an analytic equivalence relation, \( \Sigma^1_1 \) in parameter \( x_0 \). Then either there exists a perfect set of pairwise \( E \)-inequivalent reals or every \( E \)-equivalence class has a representative in a set-generic extension of \( L[x_0] \).

Hjorth’s proof makes use of his analysis of nonstandard Ehrenfeucht-Mostowski models built from #’s. Instead, we construct the necessary nonstandard models using infinitary model theory, assuming only the existence of weak compacts.

Theorem 1. Suppose that for every real \( x \) there is a weakly compact cardinal in \( L[x] \). Then the conclusion of Hjorth’s Theorem still holds.

The main lemma is the following.

Lemma 2. Suppose that there is a weakly compact cardinal in \( L[x] \), \( x \) a real. Then there is a countable nonstandard \( \omega \)-model \( M_x \) of \( ZF \) such that \( x \in M_x \) and \( L(M_x) = (L \text{ in the sense of } M_x) \) has an isomorphic copy in a set-generic extension of \( L[x_0] \), for any real \( x_0 \).

It is not known if the conclusion of Lemma 2 holds in \( ZFC \) alone, for arbitrary \( x \) (with \( ZF \) replaced by an arbitrary finite subtheory).

Proof of Theorem 1 from Lemma 2 (as in Hjorth [2]). Suppose that \( E \) is an analytic equivalence relation, \( \Sigma^1_1 \) in the parameter \( x_0 \), and choose an \( x_0 \)-recursive tree \( T \) on \( \omega \times \omega \times \omega^\omega \) such that \( xEy \iff T(x, y) \) has a branch. For each countable ordinal \( \alpha \) we define \( xE_\alpha y \iff \text{rank}(T(x, y)) \) is at least \( \alpha \); then \( E_\alpha \) is Borel in \( (x_0, c) \) where \( c \) is any real coding \( \alpha \) and \( E \) is the intersection of the \( E_\alpha \)’s. We may assume that each \( E_\alpha \) is an equivalence relation (see Theorem 1.4 of Hjorth [2]). A theorem of Harrington and Silver says that a \( \Pi^1_1 \)-equivalence relation has a perfect set of pairwise inequivalent reals or each equivalence class has a representative constructible from the parameter defining the relation. As \( E_\alpha \) is Borel in \( (x_0, c) \) where \( c \) is a...
real coding $\alpha$, and as we may assume that $E$ and hence each $E_\alpha$ has no perfect set of pairwise inequivalent reals, we know that each $E_\alpha$-equivalence class has a representative in $L[x_0, c]$ where $c$ is any real coding $\alpha$.

Now let $x$ be arbitrary and by Lemma 2 choose a countable nonstandard $\omega$-model $M_x$ of $ZF$ containing $(x_0, x)$ such that $L(M_x)$ has an isomorphic copy in a set-generic extension $N$ of $L[x_0]$. Let $a \in \text{ORD}(M_x)$ be nonstandard and let $c$ be a code for $a$, generic over $M_x$; then by applying Harrington-Silver in $M_x[c]$ we conclude that there is $y$ in $L(M_x)[x_0, c]$ such that $y E_\alpha x$. By choosing $c$ to be generic over $N$ as well we get that $y$ belongs to a set-generic extension of $L[x_0]$. Finally, $yE_\alpha x$ since, if not, $y E_\alpha x$ would fail for some $\alpha$ admissible in $(y, x)$ and hence for some (standard) $\alpha < a$. 

To prove Lemma 2 we discuss infinitary logic. Fix a real $x$ and assume $V = L[x]$. Let $\kappa$ be weakly compact and introduce the language $L$ consisting of the formulas in the language of set theory with constants $a$ for $a \in L_\kappa[x]$, closed under conjunctions and disjunctions of size less than $\kappa$ (however we allow a formula to have only finitely many free variables). Let $T$ be the theory of $(L_\kappa[x], a)$, $a \in L_\kappa[x]$, in this language. An $n$-type is a set of formulas $\Gamma$ with free variables $v_1 \ldots v_n$, and such a $\Gamma$ is consistent with $T$ if there is a model of $T$ and $n_1 \ldots n_m$ in $M$ such that $M |= \varphi(m_1 \ldots m_n)$ for each $\phi \in \Gamma$, where $M$ exists in a set-generic extension of $V = L[x]$. $\Gamma$ is complete if for every $\varphi(v_1 \ldots v_n)$ either $\varphi$ or $\sim \varphi$ belongs to $\Phi$.

Now work in the Lévy collapse $L[x, c]$, where $c$ is a real coding $\kappa^+$ of $L[x]$. Let $d_1, d_2, \ldots$ be $\omega$-many new constant symbols and for $D \subseteq \{d_1, d_2, \ldots\}$ let the language $L_D$ be defined like $L$ but with the new constant symbols from $D$. Define $T_0 = T \subseteq T_1 \subseteq \ldots$ and $T_D = \phi \subseteq D_1 \subseteq D_2 \subseteq \ldots$ inductively as follows: if $T_n, D_n$ have been defined select a complete $k$-type $\Gamma_n(v^\bar{v})$ in $L[x]$ consistent with $T_n$, choose $D_n \subseteq D_{n+1}$ so that $\text{card}(D_{n+1} - D_n) = k$, and let $T_{n+1} = T_n \cup \Gamma_n(\bar{d})$, where $\bar{d}$ enumerates $D_{n+1} - D_n$. This can be done in such a way that $T_n = T^*$ is $L[x]$-saturated: if $\Gamma(v^\bar{v})$ is an $L[x]$-type, $\bar{d}$ a finite sequence from $D$, and $\Gamma(\bar{d}, \bar{e})$ consistent with $T^*$, then $T^*$ includes $\Gamma(\bar{d}, \bar{c})$ for some $\bar{c}$. And note that each $T_n$ belongs to $L[x]$ (though of course $T^*$ itself makes use of the Lévy collapse $c$).

Let $M_x$ be the model determined by $T^*$, whose universe consists of (equivalence classes of) the constants $d_n, n \in \omega$. Note that a set in $L[x]$ of sentences in some $L_D$ is consistent iff each subset of $L[x]$-cardinality $< \kappa$ is, by $\Pi_1$-reflection and the equivalence of consistency with existence of a model after performing a Lévy collapse of $\kappa$. An easy consequence is that $M_x$ is nonstandard with standard ordinal $\kappa$.

Now consider $L(M_x)$: every $n$-type in the language $L_0 = (L$ as $L$ but restricted to $L_\kappa$) that is realized in $L(M_x)$ belongs to $L$, as each of its initial segments (obtained by restricting to some $L_\alpha, \alpha < \kappa$) belongs to $L$ and $\kappa$ is weakly compact. Also, just as $M_x$ is saturated for types in $L[x]$, $L(M_x)$ is saturated for types in $L$, since again by weak compactness any $L_0$-type in $L$ consistent with $T$ can be extended to a complete $\mathcal{L}$-type consistent with $T$ in $L[x]$.

Now it is clear that $L(M_x)$ has an isomorphic copy in $L[c]$ : using $c$ we can do the same construction as we did above in $L[x, c]$, obtaining $M_0$, a model that is saturated for $L_0$-types in $L$ and realizing only types in $L$. Now construct an isomorphism via a back and forth argument in $\omega$ steps between $M_0$ and $L(M_x)$.

Finally, note that by absoluteness the desired model $M_x$ exists not only in $L[x, c]$ but in $L[x]$. 


Remark. Lemma 2 can also be used to establish the following improvement of the Glimm-Effros style dichotomy theorem of Hjorth and Kechris [3]: Let $E$ be a $\Sigma^1_1$ equivalence relation. Assume that for every real $x$ there is a weakly compact cardinal in $L[x]$. Then either $E_0$ is continuously reducible to $E$, or $E$ is reducible to $2^{<\omega_1}$ by a function $\Delta^1_2$ in the codes.

References


Department of Mathematics, M.I.T., Cambridge, Massachusetts 02139
E-mail address: sdf@math.mit.edu

Equipe de Logique, University of Paris 7, 2 Place Jussieu, 75251 Paris Cedex 05, France
E-mail address: boban@logique.jussieu.fr