UPPER BOUNDS FOR THE NUMBER OF FACETS
OF A SIMPLICIAL COMPLEX

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Abstract. Here we study the maximal dimension of the annihilator ideals $0 : \Gamma \mathfrak{m}_I$ of artinian graded rings $A = P/(I, x_1^2, x_2^2, \ldots, x_d^2)$ with a given Hilbert function, where $P$ is the polynomial ring in the variables $x_1, x_2, \ldots, x_d$ over a field $K$ with each $\deg x_i = 1$, $I$ is a graded ideal of $P$, and $m$ is the graded maximal ideal of $A$. As an application to combinatorics, we introduce the notion of $j$-facets and obtain some informations on the number of $j$-facets of simplicial complexes with a given $f$-vector.

Let $P = K[x_1, x_2, \ldots, x_v]$ denote the polynomial ring in $v$ variables over a field $K$ with the standard grading, i.e., each $\deg x_i = 1$, and write $K\{\Gamma\}$ for the quotient algebra $P/(x_1^2, x_2^2, \ldots, x_v^2)$. We are interested in the dimensions of the annihilator ideals $0 : K\{\Gamma\}/I \mathfrak{m}_I$ of $K\{\Gamma\}/I$, where $m$ is the graded maximal ideal of $K\{\Gamma\}/I$. In particular, among all graded ideals $I$ of $K\{\Gamma\}$ with a given Hilbert function, we determine the maximal dimension of the socles $0 : K\{\Gamma\}/I \mathfrak{m}$ of $K\{\Gamma\}/I$. The graded ring $K\{\Gamma\}/I$ is studied in [A–H–H] when $I$ is generated by (squarefree) monomials.

First, we recall some standard notation and terminology on graded rings and modules. When $M$ is a $Z$-graded module, where $Z$ is the set of integers, we write $M_i$, $i \in Z$, for the $i$-th graded component of $M$. Moreover, for every $a \in Z$, we define $M(a)$ to be the $Z$-graded module with graded components $M(a)_i = M_{a+i}$ for all $i \in Z$. If $M$ is a finitely generated $Z$-graded module over the polynomial ring $P = K[x_1, x_2, \ldots, x_v]$, then the modules $\text{Tor}_i^K(K, M)$ are finite-dimensional graded $K$-vector spaces. Then we say that $\beta_{ij}(M) := \dim_K \text{Tor}_i^K(K, M)_j$ is the $(i, j)$-th graded Betti number of $M$. Finally, when $A$ is a graded ring over $K$ and $J$ is a graded ideal of $A$, we denote by $0 : A J$ the annihilator of $J$ in $A$.

Let $\binom{v}{q}$ denote the set of all squarefree monomials of degree $q \geq 1$ in the variables $V = \{x_1, x_2, \ldots, x_v\}$. We write $\leq_{\text{lex}}$ for the lexicographic order on $\binom{v}{q}$, i.e., if $S = x_{i_1}x_{i_2}\cdots x_{i_q}$ and $T = x_{j_1}x_{j_2}\cdots x_{j_q}$ are squarefree monomials belonging to $\binom{v}{q}$ with $1 \leq i_1 < i_2 < \cdots < i_q \leq v$ and $1 \leq j_1 < j_2 < \cdots < j_q \leq v$, then $S \leq_{\text{lex}} T$ if $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}$ and $i_k > j_k$ for some $1 \leq k \leq q$. A nonempty set $\mathcal{M} \subset \binom{v}{q}$ is called a squarefree lexsegment set of degree $q$ if $T \in \mathcal{M}$, $S \in \binom{v}{q}$ and $T \leq_{\text{lex}} S$ imply $S \in \mathcal{M}$. An ideal $I$ of $K\{\Gamma\}$ generated by squarefree monomials is

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called a \textit{squarefree lexsegment ideal} if, for every $1 \leq q \leq v$, $T \in I \cap \binom{V}{q}$, $S \in \binom{V}{q}$ and $T \leq_{\text{lex}} S$ imply $S \in I$.

We are now in the position to state our algebraic result of this paper.

\textbf{Theorem.} (a) Suppose that $I$ is a graded ideal of $K\{\Gamma\}$ with $I_0 = I_1 = (0)$. Then, there exists a unique squarefree lexsegment ideal $I_{\text{lex}}$ of $K\{\Gamma\}$ with the same Hilbert function as $I$.

(b) Let $m$ be the graded maximal ideal of $K\{\Gamma\}$. Fix $j \geq 0$. Suppose that for every $i \geq 0$ we have

$$\dim_K(I/m^j I)_i = \dim_K(I_{\text{lex}}/m^j I_{\text{lex}})_i.$$  

\begin{enumerate}
\item [Then for every $i \geq 0$ we have]
$$\dim_K(I/m^{j+1} I)_i \leq \dim_K(I_{\text{lex}}/m^{j+1} I_{\text{lex}})_i.$$
\item [Proof. First, we choose any term order $\rho$ for the monomials in the polynomial ring $P = K[x_1, x_2, \ldots, x_v]$ and we write $J \subset P$ for the preimage of $I$ under the canonical epimorphism $P \rightarrow K\{\Gamma\}$. It is well known (e.g., [M–M], [B–H–V]) that $P/J$ and $P/\text{in}_\rho(J)$ have the same Hilbert function and that we have the inequality $\beta_{ij}(P/J) \leq \beta_{ij}(P/\text{in}_\rho(J))$ for every $i$ and $j$.

We have the equalities $\beta_{ij}(P/J) = \dim_K(I/m^j I)_i$ if $j \geq 2$; $\beta_{12}(P/J) = \dim_K(I/m^1 I)_2$; and $\beta_{ij}(P/J) = \dim_K(\text{Soc}_i (P/J))$ for every $j \geq v$. The similar results hold for the ideal $\text{in}_\rho(J)$.

Since $K\{\Gamma\}/I \simeq P/J$ and $K\{\Gamma\}/I' \simeq P/\text{in}_\rho(J)$, it follows that $K\{\Gamma\}/I$ and $K\{\Gamma\}/I'$ have the same Hilbert function, and that $\dim_K(I/m^j I)_i \leq \dim_K(I'/m^j I')_i$ and $\dim_K(\text{Soc}_i(K\{\Gamma\}/I)) \leq \dim_K(\text{Soc}_i(K\{\Gamma\}/I'))$ for every $i$.

Thus, replacing $I$ with $I'$ and noting that $I'$ is generated by squarefree monomials, we may assume from the beginning that $I$ itself is generated by squarefree monomials.

Now since $I$ is an ideal in $K\{\Gamma\}$ generated by squarefree monomials, the existence (and uniqueness) of $I_{\text{lex}}$ is an immediate consequence of the Kruskal–Katona theorem which, stated in algebraic language, guarantees the following: Suppose that $L \subset K\{\Gamma\}$ is an ideal generated by squarefree monomials, all of the same degree, say $q$, and let $L_{\text{lex}}$ denote the ideal generated by the squarefree lexsegment set $M$ of degree $q$ with $\sharp(M) = \dim_K L_q$. Then $\dim_K L_{q+1} \geq \dim_K(L_{\text{lex}})_{q+1}$. Thanks to this fact, given a squarefree ideal $I$ of $K\{\Gamma\}$, if we consider for each $i$ the vector space $V_i \subset K\{\Gamma\}$ spanned by the squarefree lexsegment set $M_i$ of degree $i$ with $\sharp(M_i) = \dim_K I_i$, then $\bigoplus_{i \geq 0} V_i$ is an ideal of $K\{\Gamma\}$, which is just the desired $I_{\text{lex}}$. This construction also enables us to see that in each degree the number of generators of $I_{\text{lex}}$ is greater than or equal to that of $I$, which proves the inequalities in (b) for $j = 0$.

Now suppose that $j > 0$. Our hypothesis implies that $m^j I$ and $m^j I_{\text{lex}}$ have the same Hilbert function. Therefore, since $m^j I_{\text{lex}}$ is a lexsegment ideal, we conclude that $(m^j I)_{\text{lex}} = m^j I_{\text{lex}}$. Thus, as above, we deduce from the Kruskal–Katona theorem that in each degree the number of generators of $m^j I_{\text{lex}}$ is greater.
than or equal to that of \(m^jI\). In other words, we have \(\dim_K(m^jI/m^{j+1}I) \leq \dim_K(m^jI/m^{j+1}I)\). This completes the proof of the inequalities in (b) as desired.

The inequalities (c) will turn out to be again a consequence of the Kruskal–Katona theorem, but not quite as straightforward. Let us first consider the canonical module \(\omega_A\) of \(A = K\{\Gamma\}/I\). We refer the reader to, e.g., [B–H] for basic facts about canonical modules. Since \(K\{\Gamma\}\) is a Gorenstein ring (in fact, a complete intersection), we may represent \(\omega_A\), up to a shift, as a module of homomorphisms, that is to say, we have \(\omega_A(-v) = \text{Hom}_K(A, K\{\Gamma\})\). The module \(\text{Hom}_K(A, K\{\Gamma\})\) of homomorphisms may be naturally identified with the annihilator of \(I\) in \(K\{\Gamma\}\).

Hence, as a graded module, \(\omega_A(-v)\) may be regarded as the ideal in \(K\{\Gamma\}\) whose \(K\)-basis \(\Omega\) is given by all squarefree monomials \(T \in K\{\Gamma\}\) which annihilate \(I\).

We claim that \(\Omega\) is the set of all squarefree monomials \(T^c \in K\{\Gamma\}\) such that \(T \not\in I\), where \(T^c\) is defined by \(T^c = x_1 \cdots x_v/T\). In fact, suppose that \(T \not\in I\) and that \(T^cS \neq 0\) for some squarefree monomial \(S \in I\). Then, there exists a squarefree monomial \(U\) such that \(T^cSU = x_1 \cdots x_v = TT^c\). Hence \(T = SU\), and thus \(T \in I\), a contradiction. Conversely, since \(TT^c = x_1 \cdots x_v \neq 0\), if \(T^cI = 0\) then \(T \not\in I\) as desired.

It follows from the identification of \(\omega_A(-v)\) with the annihilator of \(I\) that \(\dim_K(\omega_A)_i = \dim_K A_{-i}\), for every \(i\). Thus, in particular, if \(B = K\{\Gamma\}/I^\text{lex}\), then the ideals \(\omega_A(-v)\) and \(\omega_B(-v)\) have the same Hilbert function.

The reader can verify easily, by the above description of the canonical module, that \(\omega_B(-v)\) is a squarefree lexsegment ideal of \(K\{\Gamma\}\). In fact, if \(M\) is a squarefree lexsegment set of degree \(q\) and if \(K\) is the complement of \(M\) in the set of all squarefree monomials of degree \(q\), then the set \(\{T^c; T \in K\}\) is again a squarefree lexsegment set of degree \(v - q\). This observation is crucial, since it implies that \(\omega_B(-v) = (\omega_A(-v))^{\text{lex}}\).

Finally we notice that, for every finite dimensional graded \(K\)-algebra \(C\) and for all integers \(i, j \geq 0\), we have \(\dim_K(C/m^j) = \dim_K(\omega_C/m^j\omega_C)^{-i}\). Thus if we apply the arguments in the proof of (b) to the ideal \(\omega_A(-v)\), then the assertion (c) follows.

Q. E. D.

By virtue of the Clements–Lindström theorem [C–L], the above Theorem can be generalized to ideals of the quotient algebra \(P/(x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n})\) with \(1 \leq a_1 \leq a_2 \leq \cdots \leq a_v\).

We now discuss the combinatorial implication of the above result. Let \(\Delta\) be a simplicial complex on the vertex set \(V = \{x_1, x_2, \ldots, x_v\}\), i.e., \(\Delta\) is a collection of subsets of \(V\) such that (i) \(\{x_i\} \in \Delta\) for every \(1 \leq i \leq v\) and (ii) if \(\sigma \in \Delta\) and \(\tau \subseteq \sigma\), then \(\tau \in \Delta\). Each element \(\sigma\) of \(\Delta\) is called a face of \(\Delta\). A facet of \(\Delta\) is a face \(\sigma\) of \(\Delta\) such that \(\tau \in \Delta\) and \(\sigma \subset \tau\) imply \(\sigma = \tau\). Let \(f_i = f_i(\Delta)\) be the number of faces \(\sigma\) of \(\Delta\) with \(\sharp(\sigma) = i + 1\), and \(n_i = n_i(\Delta)\) the number of facets \(\sigma\) of \(\Delta\) with \(\sharp(\sigma) = i + 1\). Here, \(\sharp(\sigma)\) is the cardinality of a finite set \(\sigma\). Note that \(f_{-1} = 1\) and \(n_{-1} = 0\). We say that \(f(\Delta) = (f_0, f_1, \ldots)\) is the \(f\)-vector of \(\Delta\). Let \(P = K[x_1, x_2, \ldots, x_v]\) denote the polynomial ring in \(v\) variables over a field \(K\) as before, and define \(I_\Delta\) to be the ideal of \(P\) generated by all squarefree monomials \(x_{i_1}x_{i_2}\cdots x_{i_r}, 1 \leq i_1 < i_2 < \cdots < i_r \leq v,\) with \(\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta\). The quotient algebra \(P/I_\Delta\) is called the Stanley–Reisner ring of \(\Delta\) over \(K\). We refer the reader to, e.g., [B–H], [H], [Hoc] and [Sta] for detailed information about Stanley–Reisner rings.
Thus, in particular, the 0-facets of $\Delta$ are just the facets of $\Delta$. Let $\mathcal{f}$ for which there exists a face $\tau$ such that $\mathcal{f}(\Delta) = (5, \ldots , x_i)$ arising from the simplicial complexes with a given $f$-vector.

We now introduce the concept of $j$-facets of simplicial complexes. We say that a face $\sigma$ of a simplicial complex $\Delta$ is a $j$-facet if $j$ is equal to the integer $\mathcal{g} \geq 1$ for which there exists a face $\tau$ of $\Delta$ such that $\sigma \cap \tau = \emptyset$, $\sigma \cup \tau \in \Delta$, and $\mathcal{g}(\sigma) = k$. Thus, in particular, the 0-facets of $\Delta$ are just the facets of $\Delta$. Let $n_j^i(\Delta)$ denote the number of $j$-facets $\sigma$ of $\Delta$ with $\mathcal{g}(\sigma) = i + 1$. For example, if $\Delta$ is a simplicial complex with $f(\Delta) = (5, 7, 1)$, then $n_j^i(\Delta) = n_j^i(\Delta^{\text{lex}})$ for every $i$ and $j$.

On the other hand, let $\Delta$ be a simplicial complex on the vertex set $\{x_1, x_2, x_3, x_4\}$ with the facets $\{x_1, x_2\}$ and $\{x_3, x_4\}$. Then, the facets of $\Delta^{\text{lex}}$ are $\{x_1\}, \{x_2, x_4\}$ and $\{x_3, x_4\}$. Hence, $n_0^1(\Delta) = 4$, while $n_0^1(\Delta^{\text{lex}}) = 3$. Hence, in general, the inequality $n_j^i(\Delta) \leq n_j^i(\Delta^{\text{lex}})$ cannot be true if $j \geq 1$. However, we have the following

**Corollary.** Let $\Delta$ be a simplicial complex and $\Delta^{\text{lex}}$ the lexsegment simplicial complex with the same $f$-vector as $\Delta$. Fix $j \geq 0$ and suppose that

$$n_j^0(\Delta) + n_j^1(\Delta) + \cdots + n_j^{i-1}(\Delta) = n_j^0(\Delta^{\text{lex}}) + n_j^1(\Delta^{\text{lex}}) + \cdots + n_j^{i-1}(\Delta^{\text{lex}})$$

for every $i \geq 0$. Then, we have the inequality

$$n_j^i(\Delta) \leq n_j^i(\Delta^{\text{lex}})$$

for every $i \geq 0$. \[Q. E. D.\]
Proof. Let \( A = K \{ \Gamma \} / I_\Delta \) and \( B = K \{ \Gamma \} / I_{\Delta \text{lex}} = K \{ \Gamma \} / (I_\Delta \text{lex}) \). Then

\[
\dim_K (0 : A m^j)_i = \sum_{k=0}^{j-1} n_{i-1}^k (\Delta)
\]

and

\[
\dim_K (0 : B m^j)_i = \sum_{k=0}^{j-1} n_{i-1}^k (\Delta \text{lex}).
\]

Hence, we can apply (c) of the theorem. Q. E. D.

Let \( \Delta \) be a lexsegment simplicial complex with \( f(\Delta) = (f_0, f_1, \ldots) \) and \( \Delta' \) the subcomplex of \( \Delta \) obtained by removing all facets of \( \Delta \). Then, \( \Delta' \) is again lexsegment with \( f(\Delta') = (\partial_1 (f_1), \partial_2 (f_2), \ldots) \), and, moreover, the facets of \( \Delta' \) are just the 1-facets of \( \Delta \). Hence, it follows that \( n_{i-1}^1 (\Delta) = \partial_i (f_i) - \partial_i (\partial_i+1 (f_i)) \) for every \( i \). In general, for every \( j \) and \( i \), we have the formula

\[
n_{i-1}^j (\Delta) = \partial_i (\partial_i+1 (\cdots (\partial_i+1 (f_i) \cdots (f_i+j-1) \cdots)) - \partial_i (\partial_i+1 (\cdots (\partial_i+1 (f_i+j) \cdots))).
\]

References


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