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AN ERGODIC THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

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ABSTRACT. This paper is concerned with an ergodic theorem for asymptotically nonexpansive mappings in the intermediate sense in Banach spaces.

1. INTRODUCTION AND THE MAIN RESULT

Throughout this paper X denotes a uniformly convex real Banach space, C a nonempty closed convex subset of X, and T a mapping from C into itself.

Recently the asymptotic behavior of asymptotically nonexpansive mappings has been studied by many authors (see [4], [14], [15] and [16]). There appear in the literature two definitions of an asymptotically nonexpansive mapping. One is due to Kirk [6] :

$$\limsup_{n\to\infty} \sup_{y\in K} (\|T^n x - T^n y\| - \|x-y\|) \le 0 \tag{1.1}$$

for each $x \in C$ and each bounded set $K \subset C$, and T^N is continuous for some $N \ge 1$. The other is due to Goebel and Kirk [5] : There exists a sequence $\{k_n\}$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x-y\| \text{ for } x, y \in C \text{ and } n = 0, 1, 2, \cdots.$$

Bruck, Kuczumow and Reich [4] have introduced a definition between these two: T is called *asymptotically nonexpansive in the intermediate sense* if T is continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x-y\|) \le 0 \tag{1.2}$$

for any bounded subset $K \subset C$. The purpose of the present paper is to prove the ergodic theorem for such a mapping in the class of Banach spaces in which the nonlinear mean ergodic theorem is usually set.

Here we summarize the notations used in the sequel. Denote by $F(T)$ the set of fixed points of a mapping T. The convex hull of a subset M of X is denoted by

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coM, and the closed convex hull by $clcoM$. We put

$$\Delta^{n-1} = \{\lambda = (\lambda_1, \cdots, \lambda_n) : \lambda_i \geq 0 \ (i = 1, 2, \cdots, n) \text{ and} \sum_{i=1}^{n} \lambda_i = 1\}$$

and

$$B_r = \{x \in X : \|x\| \leq r\} \text{ for } r > 0.$$

Also, $\omega_w(\{z_n\})$ denotes the set of weak subsequential limits of a sequence $\{z_n\}_{n\geq 0}$ in X.

Definition 1.1. A sequence $\{x_n\}_{n\geq 0}$ in C is called an *almost-orbit* of T if

$$\lim_{n\to\infty} \left[\sup_{m\geq 0} \|x_{n+m} - T^m x_n\| \right] = 0. \tag{1.3}$$

Definition 1.2. A sequence $\{z_n\}_{n\geq 0}$ in X is said to be *weakly almost convergent* to $z \in X$ if $\frac{1}{n}\sum_{i=0}^{n-1} z_{i+k}$ converges weakly as $n \to \infty$ to z uniformly in $k \geq 0$.

The main result in this paper is stated as follows:

Theorem 1.1. *Suppose that $T : C \to C$ is asymptotically nonexpansive in the intermediate sense with $F(T) \neq \phi$, and that $\{x_n\}$ is an almost-orbit of T. If the norm of X is Fréchet differentiable, then $\{x_n\}$ is weakly almost convergent to the unique point of $F(T) \cap clco\omega_w(\{x_n\})$.*

2. PROOF OF THEOREM

In what follows, a mapping $T : C \to C$ is assumed to be asymptotically nonexpansive in the intermediate sense with $F(T) \neq \phi$. Take $f \in F(T)$ and let K be a bounded closed convex subset of C including the set $\{f\}$. Put $D_K =$ diameter K.

The key point in proving mean ergodic theorems is to estimate the difference between $T^k(\sum_{i=1}^n \lambda_i z_i)$ and $\sum_{i=1}^n \lambda_i T^k z_i$ for $\lambda \in \Delta^{n-1}, z_1, \cdots, z_n \in K$ and $k \geq 1$, as done previously in [15, Proposition 3.1] and [8, Lemma 3]. However, we cannot make use of Bruck's inequality [3, Theorem 2.1] as used in [15] and [8], because our operator T is not Lipschitz continuous. Therefore our argument is different from theirs.

Lemma 2.1. *For $\varepsilon > 0$ there exist an integer $N_\varepsilon \geq 1$ and $\delta_{2,\varepsilon} > 0$ such that if $k \geq N_\varepsilon, z_1, z_2 \in K$ and if $\|z_1 - z_2\| - \|T^k z_1 - T^k z_2\| \leq \delta_{2,\varepsilon}$, then*

$$\|T^k(\lambda_1 z_1 + \lambda_2 z_2) - \lambda_1 T^k z_1 - \lambda_2 T^k z_2\| < \varepsilon$$

for all $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$.

Proof. Let δ be the modulus of uniform convexity of X and define a function $d : \mathbf{R}^+ \to \mathbf{R}^+$ by

$$d(t) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^t \delta(s)ds & \text{if } 0 \leq t \leq 2, \\ d(2) + \frac{1}{2}\delta(2)(t-2) & \text{if } t > 2. \end{cases}$$

It is then well known that d is a strictly increasing, continuous convex function, and that it satisfies

$$2\lambda_1\lambda_2 d(\|u - v\|) \leq 1 - \|\lambda_1 u + \lambda_2 v\| \tag{2.1}$$

for $\lambda = (\lambda_1, \lambda_2) \in \Delta^1, \|u\| \leq 1$ and $\|v\| \leq 1$.

For $\varepsilon > 0$ choose $\eta_\varepsilon > 0$ such that $\frac{D_K}{2} d^{-1}\left(\frac{2\eta_\varepsilon}{D_K}\right) < \varepsilon$ and put $\delta_{2,\varepsilon} = \min\left\{\eta_\varepsilon, \frac{D_K}{4}\right\}$. By (1.2) there exists an integer $N_\varepsilon \geq 1$ (depending on the set K) such that if $k \geq N_\varepsilon$,

$$\|T^k x - T^k y\| - \|x - y\| < \delta_{2,\varepsilon} \text{ for all } x, y \in K.$$

Let $k \geq N_\varepsilon$ and let $z_1, z_2 \in K$ with $\|z_1 - z_2\| - \|T^k z_1 - T^k z_2\| \leq \delta_{2,\varepsilon}$. It suffices to show Lemma 2.1 in the case of $0 < \lambda_i < 1$ $(i = 1, 2)$.

Put

$$u = \frac{T^k z_2 - T^k(\lambda_1 z_1 + \lambda_2 z_2)}{\lambda_1(\|z_1 - z_2\| + \delta_{2,\varepsilon})} \text{ and } v = \frac{T^k(\lambda_1 z_1 + \lambda_2 z_2) - T^k z_1}{\lambda_2(\|z_1 - z_2\| + \delta_{2,\varepsilon})}.$$

Then we have $\|u\| \leq 1, \|v\| \leq 1$ and

$$\lambda_1 u + \lambda_2 v = \frac{T^k z_2 - T^k z_1}{\|z_1 - z_2\| + \delta_{2,\varepsilon}}. \tag{2.2}$$

Since

$$u - v = \frac{\lambda_1 T^k z_1 + \lambda_2 T^k z_2 - T^k(\lambda_1 z_1 + \lambda_2 z_2)}{\lambda_1 \lambda_2(\|z_1 - z_2\| + \delta_{2,\varepsilon})}$$

and $\frac{2}{D_K}\lambda_1\lambda_2(\|z_1 - z_2\| + \delta_{2,\varepsilon}) \leq \frac{2}{D_K} \cdot \frac{1}{4} \cdot \left(D_K + \frac{D_K}{4}\right) < 1$, we have by (2.1) and (2.2)

$$\begin{aligned}
&d\left(\frac{2}{D_K}\|\lambda_1 T^k z_1 + \lambda_2 T^k z_2 - T^k(\lambda_1 z_1 + \lambda_2 z_2)\|\right) \\
&\leq \frac{2}{D_K}\lambda_1\lambda_2(\|z_1 - z_2\| + \delta_{2,\varepsilon}) d(\|u - v\|) \\
&\leq \frac{2}{D_K}\lambda_1\lambda_2(\|z_1 - z_2\| + \delta_{2,\varepsilon}) \frac{1}{2\lambda_1\lambda_2}\left\{1 - \frac{\|T^k z_1 - T^k z_2\|}{\|z_1 - z_2\| + \delta_{2,\varepsilon}}\right\} \\
&= \frac{1}{D_K}(\|z_1 - z_2\| - \|T^k z_1 - T^k z_2\| + \delta_{2,\varepsilon}) \leq \frac{2\delta_{2,\varepsilon}}{D_K} \leq \frac{2\eta_\varepsilon}{D_K}.
\end{aligned}$$

Here we have used the convexity of a function d and the fact that $d(0) = 0$. Consequently, we obtain from the choice of η_ε

$$\|T^k(\lambda_1 z_1 + \lambda_2 z_2) - \lambda_1 T^k z_1 - \lambda_2 T^k z_2\| \leq \frac{D_K}{2} d^{-1}\left(\frac{2\eta_\varepsilon}{D_K}\right) < \varepsilon. \quad \square$$

Lemma 2.2. *For each $\varepsilon > 0$ and each integer $n \geq 2$ there exist an integer $N_\varepsilon \geq 1$ and $\delta_{n,\varepsilon} > 0$, where N_ε is independent of n, such that if $k \geq N_\varepsilon, z_1, \cdots, z_n \in K$ and if $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_{n,\varepsilon}$ for $1 \leq i, j \leq n$, then*

$$\left\| T^k\left(\sum_{i=1}^n \lambda_i z_i\right) - \sum_{i=1}^n \lambda_i T^k z_i \right\| < \varepsilon$$

for all $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Delta^{n-1}$.

Proof. Let $\varepsilon > 0$ and let $n \geq 2$ be an arbitrary integer. Choose an integer $N_\varepsilon \geq 1$ in Lemma 2.1. We shall construct $\delta_{n,\varepsilon} (n = 2, 3, \cdots)$ inductively. Let $\delta_{2,\varepsilon}$ be as in Lemma 2.1. Suppose that all $\delta_{q,\varepsilon}$ are constructed for $q = 2, \cdots, p$. Let $\varepsilon' = \min\left(\frac{1}{3}\delta_{p,\frac{\varepsilon}{2}}, \frac{\varepsilon}{2}\right)$ and put $\delta_{p+1,\varepsilon} = \min(\delta_{2,\varepsilon'}, \varepsilon')$.

Let $\lambda \in \Delta^p, z_1, \cdots, z_{p+1} \in K, k \geq N_\varepsilon$ and $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_{p+1,\varepsilon}$ for $1 \leq i, j \leq p+1$. The case $\lambda_{p+1} = 1$ is trivial and so we assume $\lambda_{p+1} \neq 1$. Putting

$$u_j = (1-\lambda_{p+1})z_j + \lambda_{p+1}z_{p+1},\ \mu_j = \frac{\lambda_j}{1-\lambda_{p+1}} \text{ and}$$
$$u'_j = (1-\lambda_{p+1})T^k z_j + \lambda_{p+1}T^k z_{p+1}$$

for $j = 1, 2, \cdots, p$, we have

$$\sum_{i=1}^{p+1} \lambda_i z_i = \sum_{j=1}^{p} \frac{\lambda_j}{1-\lambda_{p+1}}\{(1-\lambda_{p+1})z_j + \lambda_{p+1}z_{p+1}\} = \sum_{j=1}^{p} \mu_j u_j \text{ and}$$
$$\sum_{i=1}^{p+1} \lambda_i T^k z_i = \sum_{j=1}^{p} \mu_j u'_j$$

and hence

$$\left\| T^k \left(\sum_{i=1}^{p+1} \lambda_i z_i \right) - \sum_{i=1}^{p+1} \lambda_i T^k z_i \right\| = \left\| T^k \left(\sum_{j=1}^{p} \mu_j u_j \right) - \sum_{j=1}^{p} \mu_j u'_j \right\| \tag{2.3}$$
$$\leq \left\| T^k \left(\sum_{j=1}^{p} \mu_j u_j \right) - \sum_{j=1}^{p} \mu_j T^k u_j \right\| + \sum_{j=1}^{p} \mu_j \|T^k u_j - u'_j\|.$$

Since $\|z_j - z_{p+1}\| - \|T^k z_j - T^k z_{p+1}\| \leq \delta_{p+1,\varepsilon} \leq \delta_{2,\varepsilon'}$, we have by Lemma 2.1

$$\|u'_j - T^k u_j\| = \|\{(1-\lambda_{p+1})T^k z_j + \lambda_{p+1}T^k z_{p+1}\} - T^k\{(1-\lambda_{p+1})z_j + \lambda_{p+1}z_{p+1}\}\| \leq \varepsilon'$$

for $1 \leq j \leq p$ and

$$\|u_j - u_l\| - \|u'_j - u'_l\| = (1-\lambda_{p+1})\{\|z_j - z_l\| - \|T^k z_j - T^k z_l\|\}$$
$$\leq (1-\lambda_{p+1})\delta_{p+1,\varepsilon} \leq \varepsilon' = \min\left(\frac{1}{3}\delta_{p,\frac{\varepsilon}{2}}, \frac{\varepsilon}{2}\right)$$

for $1 \leq j, l \leq p$. Therefore we obtain

$$\|u_j - u_l\| - \|T^k u_j - T^k u_l\| \leq \|u_j - u_l\| - \|u'_j - u'_l\| + \|u'_l - T^k u_l\| + \|u'_j - T^k u_j\|$$
$$\leq \frac{1}{3}\delta_{p,\frac{\varepsilon}{2}} + \varepsilon' + \varepsilon' \leq \delta_{p,\frac{\varepsilon}{2}}$$

for $1 \leq j, l \leq p$, and thus by the inductive assumption and (2.3) the desired conclusion holds. □

Since X is uniformly convex, it has the *convex approximation property* (C.A.P.), i.e. for each $\varepsilon > 0$ there exists an integer $p(= p(\varepsilon)) \geq 1$ such that for all subsets M in X whose diameters are uniformly bounded,

$$coM \subset co_p M + B_\varepsilon, \tag{2.4}$$

where $co_p M := \{\lambda_1 z_1 + \cdots + \lambda_p z_p : \lambda \in \Delta^{p-1},\ z_1, \cdots, z_p \in M\}$ (see [3, Theorem 1.1]).

The following lemma shows that the positive number $\delta_{n,\varepsilon}$ in Lemma 2.2 can be chosen independently of n, thanks to this property of the space X.

Lemma 2.3. *For every $\varepsilon > 0$ and every integer $n \geq 2$ there exist an integer $N_\varepsilon \geq 1$ and $\delta_\varepsilon > 0$, where both N_ε and δ_ε are independent of n, such that if $k \geq N_\varepsilon$, $z_1, \cdots, z_n \in K$ and if $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_\varepsilon$ for $1 \leq i, j \leq n$, then*

$$\left\| T^k \left(\sum_{i=1}^{n} \lambda_i z_i \right) - \sum_{i=1}^{n} \lambda_i T^k z_i \right\| < \varepsilon$$

for all $\lambda \in \Delta^{n-1}$.

Proof. Fix $\varepsilon > 0$ and an integer $n \geq 2$ arbitrarily. Denote by $N_{1,\varepsilon}$ the integer $N_{\frac{\varepsilon}{4}}$ in Lemma 2.2. By (1.2) there is an integer $N_{2,\varepsilon} \geq 1$ such that if $k \geq N_{2,\varepsilon}$, then we have

$$\|T^k x - T^k y\| - \|x - y\| < \varepsilon/4 \text{ for all } x, y \in K. \tag{2.5}$$

Put $N_\varepsilon = \max(N_{1,\varepsilon}, N_{2,\varepsilon})$. Let $\delta_{n,\varepsilon}(n = 2, 3, \cdots)$ be positive numbers determined in Lemma 2.2. As pointed out in the proof of [3, Theorem 2.1], $X \times X$ has the C.A.P. and hence we can choose an integer $p(= p(\varepsilon)) \geq 1$ such that

$$coM \subset co_p M + B_{\varepsilon/4} \times B_{\varepsilon/4}$$

for all subsets M in $X \times X$ whose diameters are uniformly bounded. Note that this integer p is independent of n. Put $\delta_\varepsilon = \delta_{p,\frac{\varepsilon}{4}}$.

Let $k \geq N_\varepsilon, z_1, \cdots, z_n \in K$, and $\|z_i - z_j\| - \|T^k z_i - T^k z_j\| \leq \delta_\varepsilon$ $(1 \leq i, j \leq n)$. Consider $M = \{[z_i, T^k z_i] \in X \times X : i = 1, 2, \cdots, n\}$. Note that there exists $r > 0$, independent of k and n, such that $\sup_{(x,y)\in M} \|(x, y)\|_{X\times X} \leq r$ because of (2.5) and $f \in F(T) \cap K$. Then for each $\lambda \in \Delta^{n-1}$ there exist $\mu \in \Delta^{p-1}$ and $i_1, \cdots, i_p \in \{1, 2, \cdots, n\}$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i z_i - \sum_{j=1}^{p} \mu_j z_{i_j} \right\| < \varepsilon/4$$

and

$$\left\| \sum_{i=1}^{n} \lambda_i T^k z_i - \sum_{j=1}^{p} \mu_j T^k z_{i_j} \right\| < \varepsilon/4.$$

Therefore we have by (2.5) and the choice of δ_ε

$$\begin{aligned} &\left\| T^k \left(\sum_{i=1}^{n} \lambda_i z_i \right) - \sum_{i=1}^{n} \lambda_i T^k z_i \right\| \\ &\leq \left\| T^k \left(\sum_{i=1}^{n} \lambda_i z_i \right) - T^k \left(\sum_{j=1}^{p} \mu_j z_{i_j} \right) \right\| + \left\| T^k \left(\sum_{j=1}^{p} \mu_j z_{i_j} \right) - \sum_{j=1}^{p} \mu_j T^k z_{i_j} \right\| \\ &+ \left\| \sum_{j=1}^{p} \mu_j T^k z_{i_j} - \sum_{i=1}^{n} \lambda_i T^k z_i \right\| < \varepsilon. \quad \square \end{aligned}$$

For each $\varepsilon > 0$ and each integer $k \geq 1$ set

$$F_\varepsilon(T^k) = \{x \in C : \|T^k x - x\| \leq \varepsilon\}.$$

Lemma 2.4. *For each $\varepsilon > 0$ there exist an integer $N(\varepsilon) \geq 1$ and $\delta (= \delta(\varepsilon)) > 0$ such that*

$$clco(F_\delta(T^k) \cap K) \subset F_\varepsilon(T^k) \cap K \tag{2.6}$$

for all $k \geq N(\varepsilon)$.

Proof. For each $\varepsilon > 0$ choose an integer $N_{\frac{\varepsilon}{2}} \geq 1$ and $\delta_{\frac{\varepsilon}{2}} > 0$ in Lemma 2.3.

Let $k \geq N(\varepsilon) := N_{\frac{\varepsilon}{2}}$ and put $\delta := (\delta(\varepsilon) =) \min\left(\frac{1}{2}\delta_{\frac{\varepsilon}{2}}, \frac{\varepsilon}{2}\right)$. Let $y \in co(F_\delta(T^k) \cap K)$, say $y = \sum_{i=1}^n \beta_i y_i$ $(y_i \in F_\delta(T^k) \cap K,\ i = 1, 2, \cdots, n\ ; \beta \in \Delta^{n-1})$. By Lemma 2.3,

$$\begin{aligned}
&\left\| T^k\left(\sum_{i=1}^n \beta_i y_i\right) - \sum_{i=1}^n \beta_i y_i \right\| \\
&\leq \left\| T^k\left(\sum_{i=1}^n \beta_i y_i\right) - \sum_{i=1}^n \beta_i T^k y_i \right\| + \sum_{i=1}^n \beta_i \|T^k y_i - y_i\| \leq \varepsilon,
\end{aligned}$$

because $\|y_i - y_j\| - \|T^k y_i - T^k y_j\| \leq \|y_i - T^k y_i\| + \|y_j - T^k y_j\| \leq 2\delta \leq \delta_{\frac{\varepsilon}{2}}$ for $1 \leq i, j \leq n$. Therefore $co(F_\delta(T^k) \cap K) \subset F_\varepsilon(T^k)$, which implies (2.6). ☐

Lemma 2.5. *Let $\{z_n\}$ be a sequence in K such that* w-$\lim_{n\to\infty} z_n = z$. *Suppose that for each $\varepsilon > 0$ there exists an integer $N(\varepsilon) \geq 1$ such that for $k \geq N(\varepsilon)$ there is an integer $N_{k,\varepsilon} \geq 1$ satisfying $\|T^k z_n - z_n\| < \varepsilon$ for all $n \geq N_{k,\varepsilon}$. Then we have $z \in F(T)$.*

Proof. We shall show that $\lim_{k\to\infty} \|T^k z - z\| = 0$. For $\varepsilon > 0$ choose an integer $N_{\frac{\varepsilon}{5}} \geq 1$ and $\delta_{\frac{\varepsilon}{5}} > 0$ in Lemma 2.3. By (1.2) there exists an integer $N_1(\varepsilon) \geq 1$ such that if $k \geq N_1(\varepsilon)$,

$$\|T^k u - T^k v\| - \|u - v\| < \frac{\varepsilon}{5} \tag{2.7}$$

for all $u, v \in K$. Put $\varepsilon' = \min\left(\frac{1}{2}\delta_{\frac{\varepsilon}{5}}, \frac{\varepsilon}{5}\right)$. By assumption we can take an integer $N(\varepsilon') \geq 1$. Let $N_2(\varepsilon) := \max(N_{\frac{\varepsilon}{5}}, N_1(\varepsilon), N(\varepsilon'))$ and let $k \geq N_2(\varepsilon)$.

Since $z \in clco\{z_n : n \geq N_{k,\varepsilon'}\}$, where $N_{k,\varepsilon'}$ is the integer determined by assumption, there exists a sequence $\left\{\sum_{i=1}^{l_n} \lambda_n^{(i)} z_{\phi_n(i)}\right\} \subset co\{z_n : n \geq N_{k,\varepsilon'}\}$ such that $\lim_{n\to\infty} \sum_{i=1}^{l_n} \lambda_n^{(i)} z_{\phi_n(i)} = z$.

Since

$$\begin{aligned}
&\|z_{\phi_n(i)} - z_{\phi_n(j)}\| - \|T^k z_{\phi_n(i)} - T^k z_{\phi_n(j)}\| \\
&\qquad \leq \|z_{\phi_n(i)} - T^k z_{\phi_n(i)}\| + \|z_{\phi_n(j)} - T^k z_{\phi_n(j)}\| \leq \delta_{\frac{\varepsilon}{5}}\ (1 \leq i, j \leq l_n)
\end{aligned}$$

by assumption, Lemma 2.3 implies

$$\left\| T^k\left(\sum_{i=1}^{l_n} \lambda_n^{(i)} z_{\phi_n(i)}\right) - \sum_{i=1}^{l_n} \lambda_n^{(i)} T^k z_{\phi_n(i)} \right\| < \frac{\varepsilon}{5}. \tag{2.8}$$

There is also $N_3(k, \varepsilon) \geq 1$ such that

$$\left\| \sum_{i=1}^{l_n} \lambda_n^{(i)} z_{\phi_n(i)} - z \right\| < \frac{\varepsilon}{5} \tag{2.9}$$

for all $n \geq N_3(k, \varepsilon)$. Since $z \in K$, the combination of the above inequalities with (2.7) gives

$$\begin{aligned}\|T^k z - z\| \leq & \left\| T^k z - T^k \left(\sum_{i=1}^{l_n} \lambda_n^{(i)} z_{\phi_n(i)} \right) \right\| \\ & + \left\| T^k \left(\sum_{i=1}^{l_n} \lambda_n^{(i)} z_{\phi_n(i)} \right) - \sum_{i=1}^{l_n} \lambda_n^{(i)} T^k z_{\phi_n(i)} \right\| \\ & + \left\| \sum_{i=1}^{l_n} \lambda_n^{(i)} \left(T^k z_{\phi_n(i)} - z_{\phi_n(i)} \right) \right\| + \left\| \sum_{i=1}^{l_n} \lambda_n^{(i)} z_{\phi_n(i)} - z \right\| < \varepsilon,\end{aligned}$$

whenever $n \geq N_3(k, \varepsilon)$. This shows that $\|T^k z - z\| < \varepsilon$ for $k \geq N_2(\varepsilon)$. □

Lemma 2.6. *Suppose that* $\{x_n\}_{n \geq 0}$ *and* $\{y_n\}_{n \geq 0}$ *are almost-orbits of* T. *Then* $\{\|x_n - y_n\|\}$ *converges as* $n \to \infty$.

Proof. Put $a_n = \sup_{m \geq 0} \|x_{n+m} - T^m x_n\|$ and $b_n = \sup_{m \geq 0} \|y_{n+m} - T^m y_n\|$ for $n \geq 0$. Then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$. By (1.2) for each $\varepsilon > 0$ and each integer $n \geq 1$ there exists an integer $N(\varepsilon, n) \geq 1$ such that if $m \geq N(\varepsilon, n)$, then $\|T^m x_n - T^m y_n\| - \|x_n - y_n\| < \varepsilon$. Since

$$\begin{aligned}\|x_{n+m} - y_{n+m}\| &\leq \|x_{n+m} - T^m x_n\| + \|T^m x_n - T^m y_n\| + \|T^m y_n - y_{n+m}\| \\ &\leq a_n + b_n + \|x_n - y_n\| + \varepsilon\end{aligned}$$

for $m \geq N(\varepsilon, n)$, letting $m \to \infty$, and then $n \to \infty$ and $\varepsilon \downarrow 0$, we have

$$\limsup_{m \to \infty} \|x_m - y_m\| \leq \liminf_{n \to \infty} \|x_n - y_n\|$$

and so the conclusion holds. □

Lemma 2.7. *Suppose that* $\{x_j^{(p)}\}_{j \geq 0} (p = 1, 2, \cdots)$ *are almost-orbits of* T *satisfying* $\sup\{\|x_j^{(p)}\| : j \geq 0, p \geq 1\} < \infty$. *Then for each* $\varepsilon > 0$ *and each integer* $n \geq 2$ *there exist positive integers* N_ε *and* $i_n(\varepsilon)$, *where* N_ε *is independent of* n, *such that*

$$\left\| T^k \left(\sum_{p=1}^{n} \lambda_p x_i^{(p)} \right) - \sum_{p=1}^{n} \lambda_p T^k x_i^{(p)} \right\| < \varepsilon$$

for all $k \geq N_\varepsilon, i \geq i_n(\varepsilon)$ *and* $\lambda \in \Delta^{n-1}$.

Proof. Take $f \in F(T)$ and set $K = clco(\{x_j^{(p)} : j \geq 0, p \geq 1\} \cup \{f\})$. For $\varepsilon > 0$ take an integer $N_\varepsilon \geq 1$ and $\delta_\varepsilon > 0$ in Lemma 2.3.

Since $\{\|x_j^{(p)} - x_j^{(q)}\|\}_{j \geq 0}$ converges as $j \to \infty$ by Lemma 2.6, for each p, q there exists an integer $i_0(\varepsilon, p, q) \geq 1$ such that $\|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\| < \delta_\varepsilon/3$ if $i \geq i_0(\varepsilon, p, q)$ and $k \geq 0$. Moreover there is an integer $i_1(\varepsilon, p) \geq 1$ such that $a_i^{(p)} < \delta_\varepsilon/3$ for all $i \geq i_1(\varepsilon, p)$, where $a_i^{(p)} = \sup_{j \geq 0} \|x_{i+j}^{(p)} - T^j x_i^{(p)}\|$ for $i \geq 0$.

Put $i_n(\varepsilon) = \max\{i_0(\varepsilon, p, q), i_1(\varepsilon, p) : 1 \leq p, q \leq n\}$. If $i \geq i_n(\varepsilon)$ and $k \geq N_\varepsilon$, then

$$\begin{aligned}&\|x_i^{(p)} - x_i^{(q)}\| - \|T^k x_i^{(p)} - T^k x_i^{(q)}\| \\ &\leq \|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\| + a_i^{(p)} + a_i^{(q)} < \delta_\varepsilon\end{aligned}$$

for $1 \leq p, q \leq n$ and so Lemma 2.3 gives the desired conclusion. □

By Lemma 2.6 an almost-orbit $\{x_n\}$ of T is bounded, because of $F(T) \neq \phi$.

In what follows take $f \in F(T)$ and set $K = clco(\{x_i : i \geq 0\} \cup \{f\})$ for an almost-orbit $\{x_n\}$ of T.

Lemma 2.8. *Suppose that $\{x_i\}_{i\geq 0}$ is an almost-orbit of T. Then for each $\varepsilon > 0$ there exists an integer $N_\varepsilon \geq 1$ such that for each $k \geq N_\varepsilon$, there is an integer $N_{k,\varepsilon} \geq 1$ satisfying*

$$\frac{1}{n}\sum_{i=0}^{n-1} x_{i+l} \in F_\varepsilon(T^k) \quad \text{for all } n \geq N_{k,\varepsilon} \text{ and } l \geq 0. \tag{2.10}$$

Proof. Let $\varepsilon > 0$. By Lemma 2.4 there exist an integer $N_1(\varepsilon) \geq 1$ and $\delta_1 (= \delta_1(\varepsilon)) > 0$ such that

$$clco(F_{\delta_1}(T^k) \cap K) \subset F_{\frac{\varepsilon}{3}}(T^k) \cap K \tag{2.11}$$

for $k \geq N_1(\varepsilon)$. Let $\delta(= \delta(\varepsilon)) = \min\left(\frac{\varepsilon}{12D_K}, \delta_1\right)$.

Also, from (1.2) we can choose an integer $N_2(\varepsilon) \geq 1$ such that if $k \geq N_2(\varepsilon)$,

$$\|T^k u - T^k v\| - \|u - v\| < \varepsilon/3 \tag{2.12}$$

for all $u, v \in K$.

Moreover by Lemma 2.7 for any integer $p \geq 1$ there are positive integers $N_3(\varepsilon)$ and $i_p(\varepsilon)$, where $N_3(\varepsilon)$ is independent of p, such that

$$\left\| T^k\left(\frac{1}{p}\sum_{j=0}^{p-1} x_{i+j+l}\right) - \frac{1}{p}\sum_{j=0}^{p-1} T^k x_{i+j+l} \right\| < \frac{\delta^2}{8} \tag{2.13}$$

for $k \geq N_3(\varepsilon)$, $i \geq i_p(\varepsilon)$ and $l \geq 0$.

Put $N_\varepsilon = \max\{N_i(\varepsilon) : 1 \leq i \leq 3\}$ and fix $k \geq N_\varepsilon$.

Choose $p(= p(k,\varepsilon)) \geq 1$ so that $\frac{D_K k}{p} \leq \frac{\delta^2}{2}$. Since $\{x_i\}_{i\geq 0}$ is an almost-orbit of T, there exists an integer $N_4(\varepsilon) \geq 1$ such that $\sup_{q\geq 0} \|x_{m+q} - T^q x_m\| < \frac{\delta^2}{8}$ for $m \geq N_4(\varepsilon)$. Set $w_i = \frac{1}{p}\sum_{j=0}^{p-1} x_{i+j}$ for $i \geq 0$.

If $i \geq i_p(\varepsilon) + N_4(\varepsilon)$, by (2.13) we have

(2.14)

$$\begin{aligned}
&\|w_{i+k+l} - T^k w_{i+l}\| \\
&\leq \left\| \frac{1}{p}\sum_{j=0}^{p-1}(x_{i+j+k+l} - T^k x_{i+j+l}) \right\| + \left\| \frac{1}{p}\sum_{j=0}^{p-1} T^k x_{i+j+l} - T^k\left(\frac{1}{p}\sum_{j=0}^{p-1} x_{i+j+l}\right) \right\| < \frac{\delta^2}{4}
\end{aligned}$$

for all $l \geq 0$. We also have from (2.12)

(2.15)

$$\begin{aligned}
\|w_{i+k} - T^k w_i\| &\leq \|w_{i+k} - f\| + \|T^k f - T^k w_i\| \\
&\leq \|w_{i+k} - f\| + \|f - w_i\| + \frac{\varepsilon}{3} \leq 2D_K + \frac{\varepsilon}{3} := D_1(\varepsilon)
\end{aligned}$$

for $i \geq 0$.

Choose $N_5(k,\varepsilon) \geq i_p(\varepsilon) + N_4(\varepsilon) + 1$ such that $\frac{D_1(\varepsilon)\{i_p(\varepsilon)+N_4(\varepsilon)\}}{n} < \frac{\delta^2}{4}$ for all $n \geq N_5(k,\varepsilon)$. If $n \geq N_5(k,\varepsilon)$, it then follows from (2.14) and (2.15) that

(2.16)

$$\begin{aligned}
&\frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+l} - T^k w_{i+l}\| \\
&\leq \frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+l} - w_{i+k+l}\| + \frac{1}{n}\left(\sum_{i=0}^{i_p+N-1} + \sum_{i=i_p+N}^{n-1}\right)\|w_{i+k+l} - T^k w_{i+l}\| \\
&\leq \frac{D_K k}{p} + \frac{(i_p+N)D_1(\varepsilon)}{n} + \frac{\delta^2}{4} \leq \delta^2
\end{aligned}$$

for all $l \geq 0$, where $N = N_4(\varepsilon)$ and $i_p = i_p(\varepsilon)$. Finally, choose $N_6(k,\varepsilon) \geq 1$ so that $\frac{(p-1)D_K}{2n} < \frac{\varepsilon}{12}$ for all $n \geq N_6(k,\varepsilon)$. Put $N_{k,\varepsilon} = \max(N_5(k,\varepsilon), N_6(k,\varepsilon))$. Let $n \geq N_{k,\varepsilon}$ and $l \geq 0$.

Set $A(k,n,l) = \{i \in Z : 0 \leq i \leq n-1 \text{ and } \|w_{i+l} - T^k w_{i+l}\| > \delta\}$ and $B(k,n,l) = \{0,1,\cdots,n-1\} \setminus A(k,n,l)$.

By (2.16), $\sharp A(k,n,l) \leq n\delta$, where $\sharp$ denotes cardinality. Since $\delta D_K < \frac{\varepsilon}{12}$ and $\frac{(p-1)D_K}{2n} < \frac{\varepsilon}{12}$, by (2.11) we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=0}^{n-1}x_{i+l} = \frac{1}{n}\sum_{i=0}^{n-1}w_{i+l} + \frac{1}{np}\sum_{i=1}^{p-1}(p-i)(x_{i+l-1} - x_{i+l+n-1}) \\
&= \left[\frac{1}{n}(\sharp A(k,n,l))\cdot f + \frac{1}{n}\sum_{i\in B(k,n,l)} w_{i+l}\right] + \left[\frac{1}{n}\sum_{i\in A(k,n,l)}(w_{i+l} - f)\right] \\
&+ \frac{1}{np}\sum_{i=1}^{p-1}(p-i)(x_{i+l-1} - x_{i+l+n-1}) \\
&\in clco(F_\delta(T^k)\cap K) + B_{\frac{\varepsilon}{12}} + B_{\frac{\varepsilon}{12}} \subset (F_{\frac{\varepsilon}{3}}(T^k)\cap K) + B_{\frac{\varepsilon}{6}}
\end{aligned}$$

for all $l \geq 0$. Combining this with (2.12) we find the desired claim. □

Lemma 2.9. *Suppose that the norm of X is Fréchet differentiable and that $\{x_n\}$ is an almost-orbit of T. Then the following hold:*

(i) *$\{\langle x_n, J(f-g)\rangle\}$ converges as $n \to \infty$ for every $f, g \in F(T)$, where J is the normalized duality map of X.*

(ii) *$F(T) \cap clco\ \omega_w(\{x_n\})$ is at most a singleton.*

Proof. Let $\lambda \in (0,1)$ and $f, g \in F(T)$. By Lemma 2.7 and (1.2) for $\varepsilon > 0$ there exist an integer $N_\varepsilon \geq 1$ and $i_2(\varepsilon) \geq 1$ such that if $k \geq N_\varepsilon$ and $n \geq i_2(\varepsilon)$,

$$\begin{aligned}
&\|T^k(\lambda x_n + (1-\lambda)f) - \lambda T^k x_n - (1-\lambda)f\| < \varepsilon, \\
&\|T^k u - T^k v\| - \|u - v\| < \varepsilon
\end{aligned}$$

for all $u, v \in K$. Then we have

$$\begin{aligned}
&\|\lambda x_{n+m} + (1-\lambda)f - g\| \leq \lambda\|x_{n+m} - T^m x_n\| \\
&+ \|\lambda T^m x_n + (1-\lambda)f - T^m(\lambda x_n + (1-\lambda)f)\| + \|\lambda x_n + (1-\lambda)f - g\| + \varepsilon \\
&\leq \sup_{l\geq 0}\|x_{n+l} - T^l x_n\| + \|\lambda x_n + (1-\lambda)f - g\| + 2\varepsilon
\end{aligned}$$

for $m \geq N_\varepsilon$ and $n \geq i_2(\varepsilon)$. Letting $m \to \infty$, and then $n \to \infty$ and $\varepsilon \downarrow 0$, we get

$$\limsup_{m\to\infty} \|\lambda x_m + (1-\lambda)f - g\| \leq \liminf_{n\to\infty} \|\lambda x_n + (1-\lambda)f - g\|$$

and so $\|\lambda x_n + (1-\lambda)f - g\|$ converges as $n \to \infty$. The claim (ii) is shown by the same way as in [7, Lemma 3.6]. □

Using Lemmas 2.5, 2.8 and 2.9, by the same argument used in the proof of [8, Theorem] we can easily prove Theorem 1.1.

We also have the following result on the weak convergence of an almost-orbit $\{x_n\}$ of T.

Theorem 2.10. *Let $\{x_n\}_{n\geq 0}$ be an almost-orbit of T. Suppose that*

$$\text{w-}\lim_{n\to\infty}(x_n - x_{n+1}) = 0.$$

Then we have $\omega_w(\{x_n\}) \subset F(T)$. Further if the norm of X is Fréchet differentiable, $\{x_n\}$ converges weakly to an element in $F(T)$.

Proof. Let $u \in \omega_w(\{x_n\})$. Then by definition there is a subsequence $\{p_m\}$ of $\{p\}$ such that $\text{w-lim}_{m\to\infty} x_{p_m} = u$. We have from assumption $\text{w-lim}_{m\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} x_{i+p_m} = \text{w-}\lim_{m\to\infty} x_{p_m} = u$ for any fixed $n \geq 1$. Thus Lemmas 2.5 and 2.8 show $u \in F(T)$. The last assertion follows from Lemma 2.9 (ii). □

Remark 2.1. (i) By using the same argument as in [9] with the aid of Lemmas 2.5, 2.7 and 2.8 we see that in Theorem 1.1 the assumption that the norm of X is Fréchet differentiable may be replaced by the assumption that X satisfies Opial's property.

(ii) We can also prove more general ergodic theorems corresponding to [10, Theorems 1 and 2] for semigroups of asymptotically nonexpansive mappings in the intermediate sense. As a consequence we see that [14, Corollary 4] holds for asymptotically nonexpansive mappings in the intermediate sense.

(iii) We do not know whether Theorem 2.10 is valid for asymptotically nonexpansive mappings satisfying (1.1).

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