PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 6, June 1997, Pages 1711–1716
S 0002-9939(97)03753-2

BEALS-CORDES-TYPE CHARACTERIZATIONS OF PSEUDODIFFERENTIAL OPERATORS

MICHAEL E. TAYLOR

(Communicated by Christopher D. Sogge)

Abstract. We show that, if U is the representation of $SO_e(n+1,1)$ on $L^2(S^n)$ given by (2.11), and P is a bounded operator on $L^2(S^n)$, then P belongs to $OPS^0_{1,0}(S^n)$ if and only if

$$P(g) = U(g)PU(g)^{-1}$$

is a C^∞ function on $SO_e(n+1,1)$ with values in the Banach space $\mathcal{L}(L^2(S^n))$.

Introduction

Let M be a compact C^∞ manifold. Denote by $OPS^m_{1,0}(M)$ the space of pseudodifferential operators on M, whose symbols in local coordinates satisfy

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|},$$

and denote by $OPS^m(M)$ the subspace of classical pseudodifferential operators, whose symbols have the asymptotic behavior

$$p(x,\xi) \sim p_m(x,\xi) + p_{m-1}(x,\xi) + \cdots, \qquad |\xi| \to \infty,$$

with $p_{m-j}(x,\xi)$ homogeneous of degree $m-j$ in ξ.

It was demonstrated in [B], [C], and [D] that, if $P\colon L^2(M) \to L^2(M)$, then P belongs to $OPS^0_{1,0}(M)$ if and only if, for arbitrary $A_j \in OPS^1(M)$, and any $N \in \mathbb{Z}^+$,

$$\operatorname{ad} A_N \cdots \operatorname{ad} A_1 \cdot P\colon L^2(M) \to L^2(M),$$

where $\operatorname{ad} A_j \cdot B = [A_j, B]$.

It is desirable to have some alternative characterizations, and we give one in §2 of this paper, which has as a special case the following (when $M = S^n$, the n-dimensional sphere):

Theorem. *If U is the representation of $SO_e(n+1,1)$ on $L^2(S^n)$ given by (2.11), and P is a bounded operator on $L^2(S^n)$, then P belongs to $OPS^0_{1,0}(S^n)$ if and only if*

$$P(g) = U(g)PU(g)^{-1}$$

is a C^∞ function on $SO_e(n+1,1)$ with values in the Banach space $\mathcal{L}(L^2(S^n))$.

Received by the editors July 5, 1995 and, in revised form, December 6, 1995.
1991 *Mathematics Subject Classification.* Primary 35S05.
This work was partially supported by the National Science Foundation.

It is technically useful that this characterization involves smoothness on a finite dimensional Lie group. One implication is that the group of invertible operators in $OPS^0_{1,0}(M)$ forms a smooth tame Frechét Lie group (in the terminology of [H]), by arguments such as used in [P]. This in turn leads to a simplification and elucidation of some of the pseudodifferential operator estimates in [GY]. We plan to pursue this matter further, in another paper.

1. Generalities; $\mathfrak{A}$-smooth operators

Let $\mathfrak{A} = \{A_j\}$ be a collection of operators in $OPS^1(M)$, for some compact smooth Riemannian manifold M. We will make the following two hypotheses on $\mathfrak{A}$:

$$A_j^* = A_j, \quad \mathrm{mod}\, OPS^0(M); \tag{1.1}$$

$$\text{some finite sum } L = \sum A_j^2 \text{ is elliptic in } OPS^2(M). \tag{1.2}$$

An operator $P \in \mathcal{L}(L^2(M))$ will be called $\mathfrak{A}$-smooth provided

$$(\mathrm{ad}\, A)^J P \text{ is bounded on } L^2(M), \tag{1.3}$$

for each $(\mathrm{ad}\, A)^J = \mathrm{ad}\, A_{j_l} \cdots \mathrm{ad}\, A_{j_1}$, where $\mathrm{ad}\, A \cdot P = [A, P]$. We use a common multi-index notation: $J = (j_l, \dots, j_1)$, we also set $|J| = l$. A priori, the hypothesis $P \colon L^2(M) \to L^2(M)$ implies that $(\mathrm{ad}\, A)^J P \colon C^\infty(M) \to \mathcal{D}'(M)$.

We mention a few very simple results, with almost trivial proofs.

Lemma 1.1. *The set $\mathcal{S}_\mathfrak{A}$ of $\mathfrak{A}$-smooth operators is an algebra, containing $OPS^0_{1,0}(M)$.*

Proof. That $\mathcal{S}_\mathfrak{A}$ is an algebra follows from the identity

$$(\mathrm{ad}\, A_j)(P_1 P_2) = P_1(\mathrm{ad}\, A_j \cdot P_2) + (\mathrm{ad}\, A_j \cdot P_1) P_2. \tag{1.4}$$

□

Lemma 1.2. *Let $\mathfrak{A}' = \{A_j + B_j\}$, $B_j \in OPS^0(M)$. If P is $\mathfrak{A}$-smooth, it is also $\mathfrak{A}'$-smooth.*

Lemma 1.3. *If P is $\mathfrak{A}$-smooth, so is P^*.*

Proof. We have $\mathrm{ad}\, A_j \cdot P^* = -(\mathrm{ad}\, A_j^* \cdot P)^*$, hence

$$(\mathrm{ad}\, A)^J P^* = (-1)^{|J|} [(\mathrm{ad}\, A^*)^J P]^*. \tag{1.5}$$

□

Lemma 1.4. *If P is $\mathfrak{A}$-smooth, so is each $(\mathrm{ad}\, A)^J P$.*

Lemma 1.5. *If P is $\mathfrak{A}$-smooth, then P and $(\mathrm{ad}\, A)^J P$ preserve each Sobolev space $H^s(M)$.*

Proof. We have

$$A_k P = P A_k + \mathrm{ad}\, A_k \cdot P.$$

Since the left side maps $L^2(M)$ to $H^{-1}(M)$, we have

$$P A_k \colon L^2(M) \to H^{-1}(M).$$

By (1.2) it follows that $P \colon H^{-1}(M) \to H^{-1}(M)$. That $P \colon H^1(M) \to H^1(M)$ follows by duality, given Lemma 1.3. The rest follows by iteration and interpolation.

□

We next note the approximate commutativity of P with L. Indeed,

$$[L,P]=\sum\{(\operatorname{ad}A_j\cdot P)A_j+A_j(\operatorname{ad}A_j\cdot P)\}. \tag{1.6}$$

If P is $\mathfrak{A}$-smooth, this maps $H^s(M)$ to $H^{s-1}(M)$. Hence

$$\begin{aligned} LPL^{-1}-P&=[L,P]L^{-1},\\ L^{-1}PL-P&=L^{-1}[P,L], \end{aligned} \tag{1.7}$$

with each right side mapping $H^s(M)$ into $H^{s+1}(M)$.

If P is $\mathfrak{A}$-smooth, it can often be established that P is $\widetilde{\mathfrak{A}}$-smooth, for a much bigger collection of operators $\widetilde{\mathfrak{A}}$. We make some simple comments on this phenomenon here; more incisive results will be given in §2. The next result follows immediately from Jacobi's identity

$$\operatorname{ad}A_1\operatorname{ad}A_2-\operatorname{ad}A_2\operatorname{ad}A_1=\operatorname{ad}[A_1,A_2]. \tag{1.8}$$

Lemma 1.6. *Let $\mathfrak{LA}$ be the Lie algebra generated by $\mathfrak{A}$. If P is $\mathfrak{A}$-smooth, then it is $\mathfrak{LA}$-smooth.*

Generalizing the class $\mathcal{S}_{\mathfrak{A}}$ of $\mathfrak{A}$-smooth operators let us say, for an operator P acting on $\mathcal{D}'(M)$ and preserving $C^\infty(M)$, that

$$P\in\mathcal{S}^m_{\mathfrak{A}}\Leftrightarrow(\operatorname{ad}A)^JP\colon H^s(M)\to H^{s-m}(M), \tag{1.9}$$

for all commutators as in (1.3). Thus $\mathcal{S}^0_{\mathfrak{A}}$ is what is denoted $\mathcal{S}_{\mathfrak{A}}$ above. Parallel to Lemma 1.1, we clearly have

$$P_j\in\mathcal{S}^{m_j}_{\mathfrak{A}}\Rightarrow P_1P_2\in\mathcal{S}^{m_1+m_2}_{\mathfrak{A}}, \tag{1.10}$$

and

$$OPS^m_{1,0}(M)\subset\mathcal{S}^m_{\mathfrak{A}}. \tag{1.11}$$

In particular, for an operator $E\colon C^\infty(M)\to\mathcal{D}'(M)$,

$$EL\in\mathcal{S}^m_{\mathfrak{A}}\Rightarrow E\in\mathcal{S}^{m-2}_{\mathfrak{A}}, \tag{1.12}$$

since $L^{-1}\in OPS^{-2}(M)$. Consequently we see that, if $\mathfrak{A}_b$ is the set of A_j over which we sum in (1.2),

$$EA_j\in\mathcal{S}^m_{\mathfrak{A}}\text{ for all }A_j\in\mathfrak{A}_b\Rightarrow E\in\mathcal{S}^{m-1}_{\mathfrak{A}}. \tag{1.13}$$

Now suppose $B_l\in OPS^0(M)$ are such that

$$B_lA_j\in\mathfrak{A}\text{ for all }A_j\in\mathfrak{A}_b. \tag{1.14}$$

Since

$$(\operatorname{ad}B_l\cdot P)A_j=\operatorname{ad}(B_lA_j)\cdot P-B_l(\operatorname{ad}A_j\cdot P), \tag{1.15}$$

we deduce that $\operatorname{ad}B_l\cdot P\in\mathcal{S}^{m-1}_{\mathfrak{A}}$ if $P\in\mathcal{S}^m_{\mathfrak{A}}$. We hence deduce the following:

Proposition 1.7. *If $(\operatorname{ad}C)^J=\operatorname{ad}C_{j_l}\cdots\operatorname{ad}C_{j_1}$, where μ of the C_j's are B_j's, satisfying* (1.14)*, and the rest are A_j's, in $\mathfrak{A}$, then*

$$P\in\mathcal{S}^m_{\mathfrak{A}}\Rightarrow(\operatorname{ad}C)^JP\in\mathcal{S}^{m-\mu}_{\mathfrak{A}}. \tag{1.16}$$

We note that this result applies when $\mathfrak{A}$ consists of all smooth vector fields on M and the B_l are multiplication operators, by smooth functions on M. In this case, Proposition 1.7 is one simple step in the well known result of Cordes (et al.) that $\mathcal{S}_{\mathfrak{A}}$ is precisely $OPS^0_{1,0}(M)$.

2. U-smooth operators; a characterization of $OPS^0_{1,0}(M)$

Let M be a compact smooth manifold, and G a Lie group, with a strongly continuous unitary representation $U(g)$ on $L^2(M)$. Let $\{X_j : 1 \le j \le m\}$ be a basis of the Lie algebra $\mathfrak{g}$. Suppose $dU(X_j) = iA_j \in OPS^1(M)$. Also suppose that, if Δ is the Laplace operator on G, with a left invariant metric, then $dU(\Delta) = L$ is an elliptic operator in $OPS^2(M)$; one could take $L = -\sum A_j^2$.

Let $P \in \mathcal{L}(L^2(M))$. We say P is U-smooth if

$$P(t) = U(t)PU(t)^{-1} \tag{2.1}$$

is a C^∞ function of $t \in G$, with values in $\mathcal{L}(L^2(M))$. If P is U-smooth, it follows that P is $\mathfrak{A}$-smooth, with $\mathfrak{A} = \{A_1, \dots, A_m\}$, i.e.,

$$\operatorname{ad} A_{j_l} \cdots \operatorname{ad} A_{j_1} \cdot P \colon L^2(M) \to L^2(M),$$

for all $j_1, \dots, j_l$. We aim to show that, under condition (2.4) given below, we have boundedness of commutators when the A_ν are replaced by arbitrary operators in $OPS^1(M)$, so that the criterion discovered by [B], [C], and [D] will apply.

Identify a small neighborhood $\mathcal{O}$ of the identity element $e \in G$ with a neighborhood of 0 in $\mathfrak{g}$, via the exponential map. Suppose $\varphi \in \mathcal{E}'(\mathcal{O})$ is a classical conormal distribution, with singularity at the origin, of the form $\varphi = \hat{f}$, with f in the symbol space $S^\mu(\mathfrak{g}^*)$. Set

$$f(A) = U(\hat{f}) = \int \hat{f}(t)\kappa(t)U(t)\,dt, \tag{2.2}$$

where $\kappa(t)\,dt$ is Haar measure (in exponential coordinates). For $\mu \ll 0$, this is readily verified to be smoothing, via the following simple result.

Lemma 2.1. *If $\varphi \in C_0^{2k}(G)$, then $U(\varphi)\colon H^s(M) \to H^{s+2k}(M)$, for all $s \in [-2k, 0]$.*

Proof. For $s = 0$, this follows from the identity

$$L^k U(\varphi) = U(\Delta^k \varphi).$$

Then we have the result for $s = -2k$ by duality, and the rest by interpolation. □

A more precise analysis of (2.2) is provided by the following result, proved in Appendix B (pp. 163–165) of [T3], following material developed in [T1] and in Chapter 12 of [T2].

Lemma 2.2. *If $f \in S^\mu(\mathfrak{g}^*)$, with $\hat{f}$ supported in $\mathcal{O}$, then $f(A) \in OPS^\mu(M)$, and the principal symbol of $f(A)$ is equal to $f(a_1, \dots, a_m)$ on $T^*M \setminus 0$, where $a_j(x,\xi)$ is the principal symbol of A_j.*

We now state our hypotheses on the action U. Consider the 'moment map'

$$\Phi(x,\xi) = (a_1(x,\xi), \dots, a_m(x,\xi)); \qquad \Phi\colon T^*M \setminus 0 \to \mathbb{R}^m \setminus 0. \tag{2.3}$$

The hypothesis is:

$$\Phi\colon T^*M \setminus 0 \to \mathbb{R}^m \setminus 0 \text{ is an imbedding.} \tag{2.4}$$

An equivalent statement is that for any $q_\mu \in S^\mu(T^*M \setminus 0)$, homogeneous of degree μ, there exists a homogeneous $f_\mu \in S^\mu(\mathfrak{g}^*)$ such that

$$q_\mu(x,\xi) = f_\mu(a_1(x,\xi), \dots, a_m(x,\xi)).$$

In light of Lemma 2.2, we see that, if $Q \in OPS^{\mu}(M)$ has principal symbol q_μ, then $Q = f_\mu(A)$, $\bmod OPS^{\mu-1}(M)$. An inductive argument easily gives the following.

Proposition 2.3. *Under hypothesis* (2.4)*, given any* $Q \in OPS^{\mu}(M)$*, there exists* $f \in S^{\mu}(\mathfrak{g}^*)$ *such that* $Q = f(A)$*,* $\bmod OPS^{-\infty}(M)$*, and such that* $\operatorname{supp} \hat{f} \subset \mathcal{O}$*.*

We now examine the commutator of an operator $f(A)$, given $f \in S^1(\mathfrak{g}^*)$, with an operator $P \in \mathcal{L}(L^2(M))$, assumed to be U-smooth. Using exponential coordinates, set

$$P(t) \sim \sum_{\beta \geq 0} P_\beta t^\beta, \qquad P_\beta = D^\beta P(0)/\beta!, \tag{2.5}$$

where $P(t)$ is given by (2.1). It easily follows that each P_β is U-smooth, if P is. Using

$$U(t)P = P(t)U(t) \sim \sum_{\beta \geq 0} P_\beta t^\beta U(t), \tag{2.6}$$

and plugging into (2.2), we get

$$f(A)P \sim Pf(A) + \sum_{\beta > 0} P_\beta \int \hat{f}(t) t^\beta \kappa(t) U(t)\, dt \sim Pf(A) + \sum_{\beta > 0} P_\beta f_\beta(A). \tag{2.7}$$

Note that

$$f \in S^1(\mathfrak{g}^*) \Rightarrow f_\beta \in S^{1-|\beta|}(\mathfrak{g}^*) \Rightarrow f_\beta(A) \in OPS^{1-|\beta|}(M).$$

Since it is clear that each element of $OPS^0_{1,0}(M)$ is U-smooth, and the set of U-smooth operators forms an algebra, we deduce the following:

Proposition 2.4. *If* P *is* U*-smooth, then for all* $f \in S^1(\mathfrak{g}^*)$*, with* $\hat{f}$ *supported in* $\mathcal{O}$*,*

$$[f(A), P] \colon L^2(M) \to L^2(M),$$

and furthermore this commutator is U*-smooth.*

It follows that, for any $f_\nu \in S^1(\mathfrak{g}^*)$, with $\hat{f}_\nu$ supported in $\mathcal{O}$, the iterated commutators

$$\operatorname{ad} f_N(A) \cdots \operatorname{ad} f_1(A) \cdot P$$

are all bounded on $L^2(M)$, if P is U-smooth. In view of Proposition 2.3, the criterion of [B], [C] applies: for any $Q_\nu \in OPS^1(M)$,

$$\operatorname{ad} Q_N \cdots \operatorname{ad} Q_1 \cdot P \colon L^2(M) \to L^2(M).$$

This gives our main conclusion:

Theorem 2.5. *Under hypothesis* (2.4)*, an operator* $P \in \mathcal{L}(L^2(M))$ *is in* $OPS^0_{1,0}(M)$ *if and only if it is* U*-smooth.*

We make some remarks on the hypothesis (2.4). Note that, for $(x, \xi) \in T^*M \setminus 0$,

$$\begin{aligned} D\Phi(x,\xi) \text{ is injective} &\Leftrightarrow \{da_j(x,\xi) \colon 1 \leq j \leq m\} \text{ spans } T_{(x,\xi)}T^*M \\ &\Leftrightarrow \{H_{a_j} \colon 1 \leq j \leq m\} \text{ spans } T_{(x,\xi)}T^*M, \end{aligned} \tag{2.8}$$

where H_{a_j} are Hamiltonian vector fields, generating an associated action $U^{\#}$ of G on $T^*M \setminus 0$. This last condition holds provided

$$G \text{ acts transitively on } T^*M \setminus 0, \text{ via } U^{\#}; \tag{2.9}$$

in fact when $T^*M\backslash 0$ is connected these conditions are equivalent. We also note that the moment map is invariantly defined as

$$\Phi\colon T^*M\backslash 0 \to \mathfrak{g}^*\backslash 0; \tag{2.10}$$

see [GS]. Furthermore, Φ intertwines the $U^{\#}$ on $T^*M\backslash 0$ with the Ad^* action of G on $\mathfrak{g}^*\backslash 0$. The condition (2.9) implies that, under the image of Φ, $T^*M\backslash 0$ covers a coadjoint orbit of G in $\mathfrak{g}^*\backslash 0$. The hypothesis (2.4) would state that Φ maps $T^*M\backslash 0$ diffeomorphically onto such a coadjoint orbit.

An important example of this phenomenon is provided by the natural action of the conformal group $SO_e(n+1,1)$ on the sphere S^n,

$$U(g)v(x) = J_g(x)^{1/2} v(g^{-1}\cdot x), \tag{2.11}$$

where $J_g(x)$ is the usual Jacobian determinant. Thus Theorem 2.5 applies in this case. Thus we have proved the Theorem stated in the introduction.

We note that this U-action on $L^2(S^n)$ is the simplest sort of principal series representation of the Lie group $SO_e(n+1,1)$. Given a general irreducible unitary representation U of a Lie group G on a Hilbert space H, one can consider the set of operators P on H such that the family (2.1) is smooth from G to $\mathcal{L}(H)$. As another sort of example, we mention the following result of [C]; see also [C2]. If G is the Heisenberg group (of dimension $2n+1$) and U the standard representation on $L^2(\mathbb{R}^n)$, then an operator P is U-smooth if and only if $P \in OPS^0_{0,0}(\mathbb{R}^n)$. It is of interest to consider families of U-smooth operators in other situations, but we will not pursue this here.

References

[B] R. Beals, *Characterization of pseudodifferential operators and applications*, Duke Math. J. **44** (1977), 45–57; correction **46** (1979), 215. MR **55:**8884; MR **80b:**47062

[C] H. Cordes, *On pseudodifferential operators and smoothness of special Lie group representations*, Manuscripta Math. **28** (1979), 51–69. MR **80m:**47047

[C2] H. Cordes, *The technique of pseudodifferential operators*, LMS #202, Cambridge Univ. Press, 1995. MR **96b:**35001

[D] J. Duneau, *Fonctions d'un operateur elliptique sur une variete compacte*, J. Math. Pures et Appl. **56** (1977), 367–391.

[GY] J. Goodman and D. Yang, *Local solvability of nonlinear partial differential equations of real principal type*, Preprint, 1987.

[GS] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge Univ. Press, 1984. MR **86f:**58054

[H] R. Hamilton, *The inverse function theorem of Nash-Moser*, Bull. AMS **7** (1982), 65–222. MR **83j:**58014

[P] K. Payne, *Smooth Frechet algebras and Lie groups of pseudodifferential operators*, Comm. Pure Appl. Math. **44** (1991), 309–337. MR **92a:**58140

[T1] M. Taylor, *Fourier integral operators and harmonic analysis on compact manifolds*, Proc. Symp. Pure Math. **35** (Part 2) (1979), 115–136. MR **81i:**58042

[T2] M. Taylor, *Pseudodifferential operators*, Princeton Univ. Press, 1981. MR **82i:**35172

[T3] M. Taylor, *Noncommutative Microlocal Analysis*, Part I, Memoirs AMS #313, 1984. MR **86f:**58156

Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599–3902

E-mail address: met@math.unc.edu