ITERATION OF COMPACT HOLOMORPHIC MAPS ON A HILBERT BALL

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Given a compact holomorphic fixed-point-free self-map, \( f \), of the open unit ball of a Hilbert space, we show that the sequence of iterates, \( (f^n) \), converges locally uniformly to a constant map \( \xi \) with \( \|\xi\| = 1 \). This extends results of Denjoy (1926), Wolff (1926), Hervé (1963) and MacCluer (1983). The result is false without the compactness assumption, nor is it true for all open balls of \( J^* \)-algebras.

1. Introduction

There has been extensive literature on the subject of iterating holomorphic functions since the early works of Julia [14], Fatou [6], [7], Denjoy [3] and Wolff [23], [24]. We refer to [2], [20] for some interesting surveys and references.

Given a fixed-point-free holomorphic map \( f: \Delta \rightarrow \Delta \) where \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \), Wolff’s theorem [24] states that there is a boundary point \( u \in \partial \Delta \) such that every closed disc internally tangent to \( \Delta \) at \( u \) is invariant under the iterates of \( f \). From this follows the result of Denjoy [3] and Wolff [23] that the iterates, \( f^n = f \circ \cdots \circ f \), of \( f \) converge to \( u \) uniformly on compact subsets of \( \Delta \). Wolff’s theorem has been extended to Hilbert balls [8], and the convergence result of Denjoy and Wolff also extends to the open unit ball of \( \mathbb{C}^n \) [13], [15], as well as some other domains in \( \mathbb{C}^n \) [1]. Nevertheless, the convergence result fails for infinite dimensional Hilbert balls and Stachura [18] has given an example to show that it fails even for biholomorphic self-maps.

Recently, Wolff-type theorems have been established for compact holomorphic self-maps of the open unit balls of \( J^* \)-algebras (which include \( C^* \)-algebras and Hilbert spaces) [6], [25]. A natural question is whether a Denjoy-Wolff-type convergence result for compact holomorphic maps on \( J^* \)-algebras might also follow from these Wolff-type theorems. We show that this is the case for Hilbert spaces, but not the case even for finite-dimensional \( C^* \)-algebras. We prove the following result.

Theorem. Let \( H \) be a Hilbert space with open unit ball \( B \). Let \( f: B \rightarrow B \) be a compact holomorphic map with no fixed point in \( B \). Then there exists \( \xi \in \partial B \) such that the sequence \( (f^n) \) of iterates of \( f \) converges locally uniformly on \( B \) to the constant map taking value \( \xi \).
We will give a simple example to show that the above result is false if $H$ is replaced by a $C^*$-algebra. We also note that it has been shown in [10] that if $f : B \to B$ is fixed-point-free and so-called firmly holomorphic, then the iterates $(f^n)$ converge pointwise to a boundary point $\xi \in \partial B$.

It may be useful to recall the Earle-Hamilton Theorem [5] which states that every holomorphic map $f : B \to B$, where $B$ is a bounded domain in a Banach space, has a fixed-point if $f(B)$ is strictly contained in $B$.

2. Preliminaries

All Banach spaces will be complex. Given bounded domains $D$ and $D'$ in any Banach spaces, we denote by $H(D, D')$ the space of all holomorphic maps $f : D \to D'$. We write $H(D)$ for $H(D, D)$. Every nonempty open ball $B$ in $D$ induces a norm $\| \cdot \|_B$ on $H(D, D')$ where $\|f\|_B = \sup_{x \in B} \|f(x)\|$ for $f \in H(D, D')$. The topology of local uniform convergence on $H(D, D')$ is the topology induced by the norms $\| \cdot \|_B$ where $B$ is an open ball in $D$ satisfying $\text{dist}(B, \partial D) > 0$, $\partial D$ being the boundary of $D$. Using Hadamard’s three circles theorem, it has been shown in [21], [22] (see also [19, Lemma 13.1]) that $\| \cdot \|_{B_1}$ and $\| \cdot \|_{B_2}$ induce the same topology for any open balls $B_1, B_2$ in $D$ satisfying $\text{dist}(B_1, \partial D) > 0$ and $\text{dist}(B_2, \partial D) > 0$. It follows that a sequence $(f_n)$ in $H(D, \overline{D})$ converges to $f \in H(D, \overline{D})$ locally uniformly if, and only if, for every $x \in D$, $(f_n)$ converges uniformly to $f$ on some open ball $B$ containing $x$ and satisfying $\text{dist}(B, \partial D) > 0$. Given any $x$ in a Banach space $X$, and $r > 0$, we let $B(x, r) = \{y \in X : \|y - x\| < r\}$. A map $f : D \to D' \subset X$ is called compact if the closure $\overline{f(D)}$ is compact in $X$.

**Lemma 1.** Let $B$ be the open unit ball of a Banach space $X$ and let $f : B \to B$ be a compact holomorphic map. Then the sequence $(f^n)$ of iterates of $f$ has a subsequence converging locally uniformly to a function in $H(B, \overline{B})$.

**Proof.** Choose a sequence $(r_n)$ in $(0, 1)$ such that $r_n \uparrow 1$ and $f(B) \cap B(0, r_1) \neq \emptyset$. We have $f(B) = \bigcup_{n=1}^{\infty} \{f(B) \cap B(0, r_n)\}$. We first find a subsequence of $(f^n)$ converging uniformly on $f(B) \cap B(0, r_1)$. By compactness of $\overline{f(B)} \cap B(0, r_1) \subset f(B)$, there is a countable set $\{z_n\}$ in $f(B) \cap B(0, r_1)$, which is dense in $\overline{f(B)} \cap B(0, r_1)$.

Since $f$ is compact, $(f^n)$ has a subsequence, $(f^{(n,1)})$, such that $(f^{(n,1)}(z_1))$ converges. Likewise, $(f^{(n,1)})$ has a subsequence $(f^{(n,2)})$ such that $(f^{(n,2)}(z_2))$ converges. Proceed to find subsequences $(f^{(n,k)})$ which converge at $z_1, \ldots, z_k$. We show that the diagonal sequence $(f^{(k,k)})$ converges uniformly on $f(B) \cap B(0, r_1)$. It suffices to show that it is uniformly Cauchy on $f(B) \cap B(0, r_1)$. Let $\varepsilon > 0$. Since $\text{dist}(B(0, r_1), \partial B) = 1 - r_1 > 0$, we have

$$\|h(z) - h(w)\| \leq \|z - w\| \frac{1}{1 - r_1}$$

for $h \in H(B)$ and $z, w \in B(0, r_1)$ (cf. [19, 1.17]). By compactness, there exist $z_{n_1}, \ldots, z_{n_l}$ in $\{z_n\}$ such that

$$\overline{f(B) \cap B(0, r_1)} \subset \bigcup_{i=1}^{l} B(z_{n_i}, \frac{\varepsilon}{3}(1 - r_1)).$$

There exists $N$ such that $j, k > N$ implies

$$\|f^{(j, j)}(z_{n_j}) - f^{(k, k)}(z_{n_k})\| < \frac{\varepsilon}{3}.$$
for $i = 1, \ldots , l$. Hence, for any $z \in f(B) \cap B(0, r_1)$, we have $z \in B(z_{n_i}, \frac{r}{2}(1 - r_1))$ for some $i$, and
\[
\|f^{(j)}(z) - f^{(k)}(z)\| \leq \|f^{(j)}(z) - f^{(j)}(z_{n_i})\| + \|f^{(j)}(z_{n_i}) - f^{(k)}(z_{n_i})\| + \|f^{(k)}(z_{n_i}) - f^{(k)}(z)\| < \frac{\varepsilon(1 - r_1)}{3(1 - r_1)} + \frac{\varepsilon(1 - r_1)}{3(1 - r_1)} + \frac{\varepsilon(1 - r_1)}{3(1 - r_1)} = \varepsilon
\]
whenever $j, k > N$. This shows that $(f^{(k)})$ is uniformly convergent on $f(B) \cap B(0, r_1)$.

We repeat the diagonal process as follows. Choose a subsequence $(f^{n_1})$ of $(f^n)$ converging uniformly on $f(B) \cap B(0, r_1)$. Then choose a subsequence $(f^{n_2})$ of $(f^{n_1})$ converging uniformly on $f(B) \cap B(0, r_2)$, and so on. The diagonal sequence $(f^{n_1})$ then converges uniformly on $f(B) \cap B(0, r_k)$ for $k = 1, 2, \ldots$.

Finally, we show that $(f^{n_1} \circ f^{n_2})$ converges locally uniformly on $B$. Pick $x \in B$ and choose $r, R > 0$ such that $r + R = 1 - \|x\|$ and $\frac{r}{R} < 1 - \|f(x)\|$. Then $B(x, r)$ and $B(f(x), \frac{r}{R})$ are contained in $B$. As in (1), dist$(B(x, r), \partial B) \geq R > 0$ implies
\[
f(B(x, r)) \subset B(f(x), \frac{r}{R}) \cap f(B) \subset B(0, r_k) \cap f(B)
\]
for some $k$. It follows that $(f^{n_k})$ converges uniformly on $f(B(x, r))$ and hence $(f^{n_1} \circ f^{n_2})$ converges uniformly on $B(x, r)$.

**Remark 1.** The above proof implies that every subsequence $(f^{n_k})$ of the iterates $(f^n)$ has a locally uniformly convergent subsequence.

We need the following version of the maximum modulus principle and we include a proof for completeness (cf. [4, p.95]).

**Lemma 2.** Let $D$ be a domain in a Banach space $X$ and let $B$ be the open unit ball of a Hilbert space $H$. Given any holomorphic function $f: D \rightarrow \overline{B}$, we have either $f(D) \subset B$ or $f(z) = \xi \in \partial B$ for all $z \in D$.

**Proof.** Suppose $f(z_0) = \xi \in \partial B$ for some $z_0 \in D$ where $D$ contains some open ball $B(z_0, r)$ with $r > 0$. We show that $f(z) = \xi$ for all $z \in D$. It suffices to show $f(v) = \xi$ for all $v \in B(z_0, r)$. Fix $v$ arbitrary in $B(z_0, r)$. Define $\varphi_v : \Delta \rightarrow \mathbb{C}$ by
\[
\varphi_v(\lambda) = \langle f(z_0 + \lambda(v - z_0)), \xi \rangle \quad (\lambda \in \Delta)
\]
where $\langle \cdot, \cdot \rangle$ denotes the inner product on $H$. Then $\varphi_v : \Delta \rightarrow \mathbb{C}$ and $\varphi_v(0) = 1$ imply $\varphi_v \equiv 1$ by the maximum modulus principle. It follows that $f(z_0 + \lambda(v - z_0)) = \xi$ for all $\lambda \in \Delta$ which gives $f(v) = \xi$ by continuity. \qed

### 3. Denjoy-Wolff-Type Result

In this section, we prove the **Theorem** and give some simple examples. Wolff’s theorem has been extended to fixed-point-free holomorphic self-maps $f$ of a Hilbert ball $B$, in which case there exists $\xi \in \partial B$ such that the “ellipsoids”
\[
E(\xi, \lambda) = \left\{ x \in B : \frac{|1 - \langle x, \xi \rangle|^2}{1 - \|x\|^2} < \lambda \right\} \quad (\lambda > 0)
\]
are invariant under $f$, and further, $\overline{E(\xi, \lambda)} \cap \partial B = \{ \xi \}$ (cf. [8]). If dim $B < \infty$, then the iterates $f^n$ must converge locally uniformly to the “Wolff point” $\xi$ (cf. [13, 15]).
Then Example 2. Suppose \( TT^*T \in Z \) whenever \( T \in Z \), where \( T^* \) denotes the adjoint of \( T \) (cf. \cite{11,19}). Every Hilbert space \( H = \mathcal{L}(\mathbb{C}, H) \) is a \( J^* \)-algebra, and so is every \( C^* \)-algebra.

Let \( B \) be the open unit ball of a \( J^* \)-algebra \( Z \) and let \( f : B \rightarrow B \) be a fixed-point-free compact holomorphic map. A Wolff-type result has been obtained in \cite{16,17} which states that under certain conditions on \( f \), there exist a "Wolff point" \( \xi \in \partial B \) and circular domains \( D_{z,\xi} (z \in B) \) invariant under \( f \) (see also \cite{25}). The question of whether the iterates \( f^n \) would converge to \( \xi \) was unanswered in \cite{16}. The following example gives a negative answer.

**Example 1.** Let \( Z = \mathbb{C} \times \mathbb{C} \) be equipped with the coordinatewise product and norm \( \| (z, w) \| = \max(|z|, |w|) \). Then \( Z \) is a \( C^* \)-algebra with open unit ball \( \Delta \times \Delta \). Pick any fixed-point-free \( h \in H(\Delta) \). Define \( f : \Delta \times \Delta \rightarrow \Delta \times \Delta \) by

\[
f(z, w) = (iz, hw) \quad (z, w \in \Delta).
\]

Then \( f \) is fixed-point-free and we have

\[
f^n(z, w) = (i^nz, h^n(w)),
\]

where \( (h^n) \) converges locally uniformly on \( \Delta \) to some \( \xi \in \partial \Delta \). The iterates \( (f^n) \) clearly do not converge to any boundary point in \( \partial(\Delta \times \Delta) \).

Nevertheless, we can still derive a Denjoy-Wolff-type convergence result for compact holomorphic maps on Hilbert spaces, by adapting MacCluer’s arguments for \( \mathbb{C}^n \) in \cite{15}. A crucial step in the proof depends on the fact that the automorphisms of a Hilbert ball map affine sets to affine sets, and consequently that the fixed-point set of a nonconstant holomorphic map is affine. In contrast, the automorphisms of the open unit ball of an arbitrary \( J^* \)-algebra may distort the affine sets and the fixed-point set of a nonconstant holomorphic map need not be affine, even in the simple case of the bidisc as shown by the example below. This is one reason why a Denjoy-Wolff-type result fails for arbitrary \( J^* \)-algebras.

Let \( B \) be the open unit ball of a Banach space \( X \). By an affine subset of \( B \) we mean a nonempty set of the form \( (c + L) \cap B \) where \( c \in X \) and \( L \) is a closed linear subspace of \( X \). If \( X \) is a Hilbert space, \( c \) can be chosen to be orthogonal to \( L \), and also, for nonconstant \( h \in H(B) \), its fixed-point set \( \text{Fix}(h) = \{ x \in B : h(x) = x \} \) is affine by \cite{12} (see also \cite{9,23.2}). This was proved in \cite{17} in finite dimensions.

**Example 2.** Let \( Z = \mathbb{C} \times \mathbb{C} \) be as in Example 1, with open unit ball \( \Delta \times \Delta \). Fix \( a \in \Delta \setminus \{0\} \). Define \( h : \Delta \times \Delta \rightarrow \Delta \times \Delta \) by

\[
h(z, w) = (g_a(w), g_{-a}(z)) \quad (z, w \in \Delta)
\]

where \( g_a(w) = \frac{a + w}{1 + aw} \). Then \( \text{Fix}(h) = \{(z, g_{-a}(z)) : z \in \Delta \} \) which is not affine since \( (0, -a) \) and \( (a, 0) \) are in \( \text{Fix}(h) \) while \( \frac{1}{2}(0, -a) + \frac{1}{2}(a, 0) \notin \text{Fix}(h) \). We also note that \( h^{2n}(z, w) = (z, w) \) and \( h^{2n+1}(z, w) = h(z, w) \). So \( (h^n(z, w))_n \) does not converge if \( (z, w) \notin \text{Fix}(h) \).

We are now ready to prove the Theorem.
Proof of the Theorem. We will use the same symbol throughout for both a constant function and its value.

Let \( \xi \in \partial B \) be the “Wolff point” of \( f \) as mentioned in the beginning of this section. Let \( \Gamma(f) \) be the set of all subsequential limits of \( \{f^n : n = 1, 2, \ldots\} \) in \( H(B, \overline{B}) \) with respect to the topology of local uniform convergence. By Lemma 1, \( \Gamma(f) \neq \emptyset \).

We first show that \( \Gamma(f) \) consists of constant maps only. Suppose, otherwise, that \( \Gamma(f) \) contains a nonconstant map \( g \in H(B, \overline{B}) \). We deduce a contradiction.

By Lemma 2, \( g(B) \subset B \). Let \( (f^{m_k}) \) be a subsequence of \( (f^n) \) converging to \( g \). Let \( m_k = n_{k+1} - n_k \). By Remark 1, \( (f^{m_k}) \) has a convergent subsequence, and we may assume, without loss of generality, that \( f^{m_k} \to h_0 \in H(B, \overline{B}) \). Since \( f^{m_k+1} = f^{m_k} \circ f^{n_k} \to h_0 \circ g \), we have \( h_0 \circ g = g \) and \( h_0 \) is the identity on \( g(B) \).

So \( h_0 \) is nonconstant, \( h_0(B) \subset B \) and \( A_0 = \text{Fix}(h_0) \) is an affine subset of \( B \). Since \( A_0 \subset \overline{f(B)} \) which is compact, it follows that \( \dim A_0 < \infty \). Clearly, \( A_0 \subset h_0(B) \).

If \( A_0 \neq h_0(B) \), then we repeat the above process. Letting \( p_k = m_{k+1} - m_k \), we may assume \( f^{p_k} \to h_1 \in H(B) \) satisfying \( h_1 \circ h_0 = h_0 \). So \( h_1 \) is the identity on \( h_0(B) \) and \( h_0(B) \subset A_1 = \text{Fix}(h_1) \). We have \( A_0 \subset A_1 \) and \( A_1 \) is a finite dimensional affine subset of \( B \), with \( \dim A_1 > \dim A_0 \). If \( A_1 \neq h_1(B) \), we repeat the process again.

Continuing in this manner, we must eventually find some \( h_i \in H(B) \) such that \( h_i(B) = A_i = \text{Fix}(h_i) \). For otherwise, we can construct a sequence \( (v_j) \subset \bigcup_{i=1}^{\infty} A_i \) with the property that \( \|v_i - v_j\| > \delta \) for all \( i \neq j \) and some \( \delta > 0 \). Since \( \bigcup_{i=1}^{\infty} A_i \subset \overline{f(B)} \) which is compact, this is clearly impossible. It follows that \( h_i^2 = h_i \).

Let \( (f^{i_k}) \) converge to \( h_i \) locally uniformly. Note that \( f(A_i) \subset A_i \). Since \( A_i \) is a finite dimensional affine subset of \( B \) and the automorphisms act transitively on \( B \), a similar argument to that in [15, p. 98] shows that \( A_i \) is biholomorphically equivalent to the open unit ball of \( \mathbb{C}^n \) where \( n = \dim A_i \). Now \( f|_{A_i} : A_i \to A_i \) is fixed-point-free and by [13, 15], there exists \( \mu \in \partial A_i \) such that \( (f|_{A_i})^n \) converges locally uniformly to \( \mu \) on \( A_i \). So \( h_i|_{A_i} = \lim_{k \to \infty} (f|_{A_i})^{i_k} = \mu \) which is impossible as \( h_i \) is nonconstant and \( h_i(B) = A_i \).

Therefore \( \Gamma(f) \) must consist of constant maps only. Now take any \( \eta \in \Gamma(f) \). Then \( \eta \in \partial B \), for otherwise \( \eta \) would be a fixed point of \( f \) in \( B \). There is a subsequence \( (f^{i_k}) \) converging to \( \eta \). Let \( \lambda > 0 \) and let \( z \in E(\xi, \lambda) \). We have

\[
\eta = \lim_{k \to \infty} f^{i_k}(z) \in \overline{E(\xi, \lambda)} \cap \partial B = \{ \xi \}
\]

since \( E(\xi, \lambda) \) is \( f \)-invariant. Therefore every convergent subsequence of \( (f^n) \) converges to the constant map \( \xi \). It follows from Remark 1 that \( (f^n) \) must converge locally uniformly to \( \xi \) and the proof is complete.

We end with the following example of a fixed-point-free compact holomorphic map on a Hilbert ball.

**Example 3.** Let \( B \) be the open unit ball of the (complex) Hilbert space \( l_2 \). Define \( f : B \to B \) by

\[
f(x_1, x_2, \ldots) = \left( \frac{1 + x_1}{2}, \frac{1 - x_1}{2} x_2, \frac{1 - x_1}{2} \frac{x_2}{3}, \ldots \right) = \left( \frac{1 + x_1}{2}, 0, 0, \ldots \right) + \frac{1 - x_1}{2} \left( 0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots \right)
\]
for \((x_1, x_2, \ldots) \in B\). Then \(f\) is fixed-point-free and holomorphic. Moreover, \(f\) is compact since it is the sum of two compact maps. Also the Wolff point is \((1, 0, 0, \ldots)\).

**Acknowledgements**

This research was carried out during the authors’ visits to each other’s institution. We gratefully acknowledge financial support from the British Council/Forbairt Joint Research Scheme and the Arts Faculty of University College Dublin.

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