

WEIGHTED HARDY-LITTLEWOOD INEQUALITY FOR A -HARMONIC TENSORS

SHUSEN DING

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ABSTRACT. In this paper we prove a local weighted integral inequality for conjugate A -harmonic tensors similar to the Hardy and Littlewood integral inequality for conjugate harmonic functions. Then by using the local weighted integral inequality, we prove a global weighted integral inequality for conjugate A -harmonic tensors in John domains.

1. INTRODUCTION AND NOTATION

Conjugate harmonic functions have wide applications in many fields, such as potential theory, harmonic analysis and the theory of H^p -spaces. Conjugate A -harmonic tensors are interesting and important generalizations of conjugate harmonic functions and p -harmonic functions, $p > 1$. Many interesting results of conjugate A -harmonic tensors and their applications in fields such as quasiregular mappings and the theory of elasticity have been found recently, see [N3], [I], [IM], [S], [B] and [BM]. In this paper, we prove local weighted inequalities and global weighted integral inequalities for conjugate A -harmonic tensors in John domains.

Let e_1, e_2, \dots, e_n denote the standard unit basis of \mathbf{R}^n . For $l = 0, 1, \dots, n$, the linear space of l -vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_l}$, corresponding to all ordered l -tuples $I = (i_1, i_2, \dots, i_l)$, $1 \leq i_1 < i_2 < \dots < i_l \leq n$, is denoted by $\wedge^l = \wedge^l(\mathbf{R}^n)$. The Grassmann algebra $\wedge = \bigoplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in \wedge is given by

$$\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$$

with summation over all l -tuples $I = (i_1, i_2, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$. We define the Hodge star operator $\star: \wedge \rightarrow \wedge$ by the rule

$$\star \mathbf{1} = e_1 \wedge e_2 \wedge \dots \wedge e_n \quad \text{and} \quad \alpha \wedge \star \beta = \beta \wedge \star \alpha = \langle \alpha, \beta \rangle (\star \mathbf{1})$$

for all $\alpha, \beta \in \wedge$.

Hence the norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star \alpha) \in \wedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on \wedge with $\star: \wedge^l \rightarrow \wedge^{n-l}$ and $\star \star (-1)^{l(n-l)}: \wedge^l \rightarrow \wedge^l$.

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Throughout this paper, we always assume Ω is a connected open subset of \mathbf{R}^n . We write $\mathbf{R} = \mathbf{R}^1$. Cubes or balls are denoted by Q and σQ is the cube or ball with the same center as Q and with $\text{diam}(\sigma Q) = \sigma \text{diam}(Q)$. The n -dimensional Lebesgue measure of a set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$. Suppose that $w \in L^1_{loc}(\mathbf{R}^n)$, $w > 0$ a.e., and $0 < p < \infty$. We denote the weighted L^p -norm of a measurable function f over E by

$$\|f\|_{p,E,w} = \left(\int_E |f(x)|^p w(x) dx \right)^{1/p}.$$

A differential l -form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(\mathbf{R}^n)$. We denote the space of differential l -forms by $D^l(\Omega, \wedge^l)$. We write $L^p(\Omega, \wedge^l)$ for the l -forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \dots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered l -tuples I . Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_{\Omega} |\omega(x)|^p dx \right)^{1/p} = \left(\int_{\Omega} \left(\sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}.$$

Similarly, $W^1_p(\Omega, \wedge^l)$ are those differential l -forms on Ω whose coefficients are in $W^1_p(\Omega, \mathbf{R})$. The notations $W^1_{p,loc}(\Omega, \mathbf{R})$ and $W^1_{p,loc}(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $d : D^l(\Omega, \wedge^l) \rightarrow D^l(\Omega, \wedge^{l+1})$ for $l = 0, 1, \dots, n$. Its formal adjoint operator $d^* : D^l(\Omega, \wedge^{l+1}) \rightarrow D^l(\Omega, \wedge^l)$ is given by $d^* = (-1)^{n(l+1)} \star d \star$ on $D^l(\Omega, \wedge^{l+1})$, $l = 0, 1, \dots, n$.

Definition 1.1. We call u a p -harmonic function if u satisfies the p -harmonic equation

$$(1.2) \quad \text{div}(\nabla u |\nabla u|^{p-2}) = 0$$

with $p > 1$. Its conjugate in the plane is a q -harmonic function v , $p^{-1} + q^{-1} = 1$, which satisfies

$$\nabla u |\nabla u|^{p-2} = \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right).$$

Note that if $p = q = 2$, we get the usual conjugate harmonic functions. If $\omega : \Omega \rightarrow \wedge^l$, then the value of $\omega(x)$ at the vectors $\xi_1, \dots, \xi_l \in \mathbf{R}^n$ will be denoted by $\omega(x; \xi_1, \dots, \xi_l)$. The following lemma appears in [IL].

Lemma 1.3. Let $Q \subset \mathbf{R}^n$ be a cube. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q, \wedge^l) \rightarrow C^\infty(Q, \wedge^{l-1})$ defined by

$$(K_y \omega)(x; \xi_1, \dots, \xi_l) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \dots, \xi_{l-1}) dt$$

and the decomposition

$$\omega = d(K_y \omega) + K_y(d\omega).$$

We define another linear operator $T_Q : C^\infty(Q, \wedge^l) \rightarrow C^\infty(Q, \wedge^{l-1})$ by averaging K_y over points y in Q

$$(1.4) \quad T_Q \omega = \int_Q \varphi(y) K_y \omega dy,$$

where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y)dy = 1$. We define the l -form $\omega_Q \in D'(Q, \wedge^l)$ by

$$(1.5) \quad \omega_Q = |Q|^{-1} \int_Q \omega(y)dy, \quad l = 0, \text{ and } \omega_Q = d(T_Q\omega), \quad l = 1, 2, \dots, n,$$

for all $\omega \in L^p(Q, \wedge^l)$, $1 \leq p < \infty$.

In recent years there has been new interest developed in the study of the A -harmonic equation for differential forms, largely pertaining to applications in quasiconformal analysis and nonlinear elasticity, that is:

$$(1.6) \quad d^*A(x, d\omega) = 0,$$

where $A : \Omega \times \wedge^l(\mathbf{R}^n) \rightarrow \wedge^l(\mathbf{R}^n)$ satisfies the following conditions:

$$(1.7) \quad |A(x, \xi)| \leq a|\xi|^{p-1} \text{ and } \langle A(x, \xi), \xi \rangle \geq |\xi|^p$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.6). A solution to (1.6) is an element of the Sobolev space $W_{p,loc}^1(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi \rangle = 0$$

for all $\varphi \in W_p^1(\Omega, \wedge^{l-1})$ with compact support.

In order to formulate the Hardy-Littlewood type estimate it is required first of all that the equation is written in the form of a first order differential system:

$$(1.8) \quad A(x, du) = d^*v.$$

In this way we obtain a pair (u, v) of $(l-1)$ -form u and $(l+1)$ -form v , called the conjugate A -harmonic fields. Example: $du = d^*v$ is an analogue of a Cauchy-Riemann system in \mathbf{R}^n . Clearly, the A -harmonic equation is not affected by adding a closed form to u and coclosed form to v . Therefore, any type of estimates between u and v must be modulo such forms. Suppose that u is a solution to (1.6) in Ω . Then by Poincaré's lemma, at least locally in a ball B , there exists a form $v \in W_q^1(B, \wedge^{l+1})$, $\frac{1}{p} + \frac{1}{q} = 1$, such that (1.8) holds.

Definition 1.9. When u and v satisfy (1.8) in Ω , and A^{-1} exists in Ω , we call u and v conjugate A -harmonic tensors in Ω .

Hardy and Littlewood in [HL] proved the following result.

Theorem A. For each $p > 0$, there is a constant C such that

$$\int_D |u - u(0)|^p dx dy \leq C \int_D |v - v(0)|^p dx dy$$

for all analytic functions $f = u + iv$ in the unit disk D .

Craig A. Nolder proves the similar results about K -quasiregular mappings in [N1] and [N2]. Recently, Craig A. Nolder generalized the above result and proved the following important Theorem B and Theorem C about conjugate A -harmonic tensors [N3].

Theorem B. *Let u and v be conjugate A -harmonic tensors in $\Omega \subset \mathbf{R}^n$, $\sigma > 1$, and $0 < s, t < \infty$. Then there exists a constant C , independent of u and v , such that*

$$(1.10) \quad \|u - u_Q\|_{s,Q} \leq C|Q|^\beta \|v - c_1\|_{t,\sigma Q}^{q/p}$$

and

$$\|v - v_Q\|_{t,Q} \leq C|Q|^{-\beta p/q} \|u - c_2\|_{s,\sigma Q}^{p/q}$$

for all cubes Q with $\sigma Q \subset \Omega$. Here c_1 is any form in $W_{p,loc}^1(\Omega, \Lambda)$ with $d^*c_1 = 0$, c_2 is any form in $W_{q,loc}^1(\Omega, \Lambda)$ with $dc_2 = 0$ and $\beta = 1/s + 1/n - (1/t + 1/n)q/p$.

Theorem C. *Let $u \in D'(\Omega, \Lambda^0)$ and $v \in D'(\Omega, \Lambda^2)$ be conjugate A -harmonic tensors and $0 < s, t < \infty$. If Ω is a δ -John domain, $q \leq p$, $v - c \in L^t(\Omega, \Lambda^2)$ and*

$$(1.11) \quad s = \Phi(t) = \frac{npt}{nq + t(q-p)}, \quad 0 < t < \infty,$$

then $u \in L^s(\Omega, \Lambda^0)$ and moreover, there exists a constant C , independent of u and v , such that

$$(1.12) \quad \|u - u_{Q_0}\|_{s,\Omega} \leq C \|v - c\|_{t,\Omega}^{q/p}.$$

Here c is any form in $W_{q,loc}^1(\Omega, \Lambda)$ with $d^*c = 0$.

Our main results Theorem 2.4 and Theorem 3.4 generalize (1.10) and (1.12), respectively.

2. THE LOCAL WEIGHTED INTEGRAL INEQUALITY

We will use the following generalized Hölder’s inequality.

Lemma 2.1. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on \mathbf{R}^n , then*

$$\|fg\|_{s,\Omega} \leq \|f\|_{\alpha,\Omega} \cdot \|g\|_{\beta,\Omega}$$

for any $\Omega \subset \mathbf{R}^n$.

The following definition appears in [G].

Definition 2.2. We say the weight $w(x) > 0$ satisfies the A_r condition, $r > 1$, write $w \in A_r$, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w dx \right) \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} < \infty$$

for any cube $Q \subset \mathbf{R}^n$.

We also need the following lemma [G].

Lemma 2.3. *If $w \in A_r$, $r > 1$, then there exist constants $\beta > 1$ and C , independent of w , such that*

$$\|w\|_{\beta,Q} \leq C|Q|^{(1-\beta)/\beta} \|w\|_{1,Q}$$

for all cubes $Q \subset \mathbf{R}^n$.

Now, we can prove the following local weighted result.

Theorem 2.4. *Let u and v be conjugate A -harmonic tensors in a domain $\Omega \subset \mathbf{R}^n$ and $w \in A_r$. Let $s = \Phi(t)$ be defined by (1.11). Then there exists a constant C , independent of u and v , such that*

$$(2.5) \quad \left(\int_Q |u - u_Q|^s w dx \right)^{1/s} \leq C \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{q/pt}$$

for all cubes Q with $\sigma Q \subset \Omega \subset \mathbf{R}^n$ and $\sigma > 1$. Here c is any form in $W_{q,loc}^1(\Omega, \Lambda)$ with $d^*c = 0$.

Proof. By Lemma 2.3, there exist constants $\alpha > 1$ and C_1 , independent of w , such that

$$(2.6) \quad \|w\|_{\alpha, \sigma Q} \leq C_1 |Q|^{(1-\alpha)/\alpha} \|w\|_{1, \sigma Q}.$$

Since $1/\alpha s + (\alpha - 1)/\alpha s = 1/s$, then by Lemma 2.1, we have

$$(2.7) \quad \|u - u_Q\|_{s, Q, w} \leq \|w\|_{\alpha, Q}^{1/s} \cdot \|u - u_Q\|_{\alpha s/(\alpha-1), Q}.$$

By Theorem B, there is a constant C_2 , independent of u and v , such that for any $t' > 0$, we have

$$(2.8) \quad \|u - u_Q\|_{\alpha s/(\alpha-1), Q} \leq C_2 |Q|^{\beta'} \cdot \|v - c\|_{t', \sigma Q}^{q/p},$$

where $\beta' = (\alpha - 1)/\alpha s + 1/n - (1/t' + 1/n)q/p$. Combining (2.7) and (2.8), we obtain

$$(2.9) \quad \|u - u_Q\|_{s, Q, w} \leq C_2 |Q|^{\beta'} \cdot \|w\|_{\alpha, Q}^{1/s} \cdot \|v - c\|_{t', \sigma Q}^{q/p}.$$

Now, choose $t' = t/k$, where k is to be determined later, and note $|v - c| = w^{-p/qs}|v - c|w^{p/qs}$; by Lemma 2.1, we get

$$(2.10) \quad \|v - c\|_{t', \sigma Q} \leq \|(1/w)^{pt/qs}\|_{1/(k-1), \sigma Q}^{1/t} \cdot \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{1/t}.$$

From (2.6), (2.9) and (2.10) we have

$$(2.11) \quad \begin{aligned} \|u - u_Q\|_{s, Q, w} &\leq C_3 |Q|^{\beta' + (1-\alpha)/\alpha s} \|w\|_{1, \sigma Q}^{1/s} \\ &\cdot \left\| \left(\frac{1}{w} \right)^{pt/qs} \right\|_{1/(k-1), \sigma Q}^{q/pt} \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{q/pt}. \end{aligned}$$

We choose $k = 1 + pt(r - 1)/qs$. Then $(k - 1)qs/pt = r - 1$, and by $w \in A_r$, we know

$$(2.12) \quad \begin{aligned} &\|w\|_{1, \sigma Q}^{1/s} \left\| \left(\frac{1}{w} \right)^{pt/qs} \right\|_{1/(k-1), \sigma Q}^{q/pt} \\ &= |\sigma Q|^{1/s + (k-1)q/pt} \left(\frac{1}{|\sigma Q|} \int_{\sigma Q} w dx \left(\frac{1}{|\sigma Q|} \int_{\sigma Q} \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} \right)^{1/s} \\ &\leq C_4 |Q|^{1/s + (k-1)q/pt}. \end{aligned}$$

By (2.11) and (2.12), we have

$$(2.13) \quad \|u - u_Q\|_{s, Q, w} \leq C_5 |Q|^\gamma \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{q/pt},$$

where $\gamma = \beta' + (1 - \alpha)/\alpha s + 1/s + q(k - 1)/pt = -(nq + t(q - p))/npt + 1/s = 0$ by (1.11). So (2.13) becomes

$$\|u - u_Q\|_{s,Q,w} \leq C \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{q/pt},$$

that is,

$$\left(\int_Q |u - u_Q|^s w dx \right)^{1/s} \leq C \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{q/pt}.$$

We have completed the proof of Theorem 2.4. □

3. THE GLOBAL WEIGHTED INTEGRAL INEQUALITY

Definition 3.1. We call Ω , a proper subdomain of \mathbf{R}^n , a δ -John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \geq \delta|x - \xi|$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

We know that a δ -John domain has the following properties [N3].

Lemma 3.2. *Let $\Omega \subset \mathbf{R}^n$ be a δ -John domain. Then there exists a covering \mathcal{V} of Ω consisting of open cubes such that:*

- i) $\sum_{Q \in \mathcal{V}} \chi_{\sigma Q}(x) \leq N\chi_{\Omega}(x), \quad x \in \mathbf{R}^n.$
- ii) *There is a distinguished cube $Q_0 \in \mathcal{V}$ (called the central cube) which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \dots, Q_k = Q$ from \mathcal{V} such that for each $i = 0, 1, \dots, k - 1,$*

$$Q \subset NQ_i.$$

There is a cube $R_i \subset \mathbf{R}^n$ (this cube does not need be a member of \mathcal{V}) such that

$$R_i \subset Q_i \cap Q_{i+1}, \text{ and } Q_i \cup Q_{i+1} \subset NR_i.$$

The following lemma appears in [IN].

Lemma 3.3. *If \mathcal{V} is a collection of cubes in \mathbf{R}^n and C_Q are non-negative numbers associated with the cubes $Q \in \mathcal{V}$ and $w \in A_r, d\mu(x) = w(x)dx,$ then for $1 \leq p < \infty$ and $N \geq 1$ we have*

$$\left(\int_{\mathbf{R}^n} \left(\sum_{Q \in \mathcal{V}} C_Q \cdot \chi_{NQ} \right)^p d\mu(x) \right)^{1/p} \leq B_p \left(\int_{\mathbf{R}^n} \left(\sum_{Q \in \mathcal{V}} C_Q \cdot \chi_Q \right)^p d\mu(x) \right)^{1/p},$$

where B_p is independent of the collection \mathcal{V} and the numbers C_Q .

Theorem 3.4. *Let $u \in D'(\Omega, \Lambda^0)$ and $v \in D'(\Omega, \Lambda^2)$ be conjugate A -harmonic tensors. Let $q \leq p, v - c \in L^t(\Omega, \Lambda^2), w \in A_r$ and $s = \Phi(t)$ is defined in (1.11). Then there exists a constant $C,$ independent of u and $v,$ such that*

$$\left(\int_{\Omega} |u - u_{Q_0}|^s w dx \right)^{1/s} \leq C \left(\int_{\Omega} |v - c|^t w^{pt/qs} dx \right)^{q/pt}$$

for any δ -John domain $\Omega \subset \mathbf{R}^n.$ Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^*c = 0$ and $Q_0 \subset \Omega$ is the cube appearing in Lemma 3.2.

Proof. Since $w \in A_r$, we can write $d\mu(x) = w(x)dx$; then (2.5) can be written as

$$(3.5) \quad \int_Q |u - u_Q|^s d\mu(x) \leq C \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{qs/pt}.$$

We use the notations and the covering \mathcal{V} described in the above Lemma 3.2 and the properties of the measure $d\mu(x) = w(x)dx$: if $w \in A_r$, then

$$(3.6) \quad \mu(NQ) \leq MN^{nr} \mu(Q)$$

for each cube Q with $NQ \subset \mathbf{R}^n$ (see [G]) and

$$(3.7) \quad \max(\mu(Q_i), \mu(Q_{i+1})) \leq MN^{nr} \mu(Q_i \cap Q_{i+1})$$

for the sequence of cubes $Q_i, Q_{i+1}, i = 0, 1, \dots, k - 1$ described in ii). We will use the elementary inequality $|a + b|^s \leq 2^s(|a|^s + |b|^s)$ for all $s > 0$. In particular we have

$$(3.8) \quad \begin{aligned} \int_{\Omega} |u - u_{Q_0}|^s w dx &= \int_{\Omega} |u - u_{Q_0}|^s d\mu(x) \\ &\leq 2^s \sum_{Q \in \mathcal{V}} \int_Q |u - u_Q|^s d\mu(x) + 2^s \sum_{Q \in \mathcal{V}} \int_Q |u_{Q_0} - u_Q|^s d\mu(x). \end{aligned}$$

The first sum can be estimated by (3.5) and the condition i):

$$(3.9) \quad \begin{aligned} \sum_{Q \in \mathcal{V}} \int_Q |u - u_Q|^s d\mu(x) &\leq C_1 \sum_{Q \in \mathcal{V}} \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{qs/pt} \\ &\leq C_1 N \left(\int_{\Omega} |v - c|^t w^{pt/qs} dx \right)^{qs/pt}. \end{aligned}$$

Now we estimate the second sum in (3.8). Fix a cube $Q \in \mathcal{V}$ and let $Q_0, Q_1, \dots, Q_k = Q$ be the chain from ii). We have

$$(3.10) \quad |u_{Q_0} - u_Q| \leq \sum_{i=0}^{k-1} |u_{Q_i} - u_{Q_{i+1}}|.$$

From (3.5) and (3.7) we have

$$\begin{aligned} |u_{Q_i} - u_{Q_{i+1}}|^s &= \frac{1}{\mu(Q_i \cap Q_{i+1})} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s d\mu(x) \\ &\leq \frac{MN^{nr}}{\max(\mu(Q_i), \mu(Q_{i+1}))} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s d\mu(x) \\ &\leq C_2 \sum_{j=i}^{i+1} \frac{1}{\mu(Q_j)} \int_{Q_j} |u - u_{Q_j}|^s d\mu(x) \\ &\leq C_3 \sum_{j=i}^{i+1} \frac{1}{\mu(Q_j)} \left(\int_{\sigma Q_j} |v - c|^t w^{pt/qs} dx \right)^{qs/pt} \end{aligned}$$

Since $Q \subset NQ_j$ for $j = i, i + 1, 0 \leq i \leq k - 1$ (see ii)), we have

$$|u_{Q_i} - u_{Q_{i+1}}|^s \chi_Q(x) \leq C_3 \sum_{j=i}^{i+1} \frac{\chi_{NQ_j}(x)}{\mu(Q_j)} \left(\int_{\sigma Q_j} |v - c|^t w^{pt/qs} dx \right)^{qs/pt}.$$

By (3.10) we have (note $|a + b|^{1/s} \leq 2^{1/s}(|a|^{1/s} + |b|^{1/s})$)

$$|u_{Q_0} - u_Q| \chi_Q(x) \leq C_4 \sum_{R \in \mathcal{V}} \left(\frac{1}{\mu(R)} \left(\int_{\sigma R} |v - c|^t w^{pt/qs} dx \right)^{qs/pt} \right)^{1/s} \cdot \chi_{NR}(x)$$

for every $x \in \mathbf{R}^n$. Hence

$$(3.11) \quad \sum_{Q \in \mathcal{V}} \int_Q |u_{Q_0} - u_Q|^s d\mu(x) \leq C_5 \int_{\mathbf{R}^n} \left| \sum_{R \in \mathcal{V}} \left(\frac{1}{\mu(R)} \left(\int_{\sigma R} |v - c|^t w^{pt/qs} dx \right)^{qs/pt} \right)^{1/s} \chi_{NR}(x) \right|^s d\mu(x).$$

If $0 \leq s \leq 1$, we use the inequality $|\sum t_\alpha|^s \leq \sum |t_\alpha|^s$, (3.6) and the condition i) to get

$$\begin{aligned} \sum_{Q \in \mathcal{V}} \int_Q |u_{Q_0} - u_Q|^s d\mu(x) &\leq C_6 \sum_{R \in \mathcal{V}} \frac{\mu(NR)}{\mu(R)} \left(\int_{\sigma R} |v - c|^t w^{pt/qs} dx \right)^{qs/pt} \\ &\leq C_7 \sum_{R \in \mathcal{V}} \left(\int_{\sigma R} |v - c|^t w^{pt/qs} dx \right)^{qs/pt}. \end{aligned}$$

Note $qs/pt \geq 1$ and $\sum t_\alpha^p \leq (\sum t_\alpha)^p$ for $p \geq 1$ and $t_\alpha > 0$; then

$$(3.12) \quad \begin{aligned} \sum_{Q \in \mathcal{V}} \int_Q |u_{Q_0} - u_Q|^s d\mu(x) &\leq C_7 \sum_{R \in \mathcal{V}} \left(\int_{\Omega} |v - c|^t w^{pt/qs} \chi_{\sigma R}(x) dx \right)^{qs/pt} \\ &\leq C_7 \left(\int_{\Omega} |v - c|^t w^{pt/qs} \sum_{R \in \mathcal{V}} \chi_{\sigma R}(x) dx \right)^{qs/pt} \\ &\leq C_7 \left(\int_{\Omega} |v - c|^t w^{pt/qs} N \chi_{\Omega}(x) dx \right)^{qs/pt} \\ &\leq C_8 \left(\int_{\Omega} |v - c|^t w^{pt/qs} dx \right)^{qs/pt}. \end{aligned}$$

Combining (3.8), (3.9) and (3.12), we have proved our theorem for the case $0 < s \leq 1$.

For the case $1 \leq s < \infty$, by (3.11) and Lemma 3.3, we have

$$\begin{aligned} \sum_{Q \in \mathcal{V}} \int_Q |u_{Q_0} - u_Q|^s d\mu(x) &\leq C_9 \int_{\mathbf{R}^n} \left| \sum_{R \in \mathcal{V}} \left(\frac{1}{\mu(R)} \left(\int_{\sigma R} |v - c|^t w^{pt/qs} dx \right)^{qs/pt} \right)^{1/s} \chi_R(x) \right|^s d\mu(x). \end{aligned}$$

Note

$$\sum_{R \in \mathcal{V}} \chi_R(x) \leq \sum_{R \in \mathcal{V}} \chi_{\sigma R}(x) \leq N \chi_{\Omega}(x)$$

with the elementary inequality $|\sum_{i=1}^N t_i|^s \leq N^{s-1} \sum_{i=1}^N |t_i|^s$; we obtain

$$\begin{aligned}
 & \sum_{Q \in \mathcal{V}} \int_Q |u_{Q_0} - u_Q|^s d\mu(x) \\
 & \leq C_{10} \int_{R^n} \left(\sum_{R \in \mathcal{V}} \frac{1}{\mu(R)} \left(\int_{\sigma R} |v - c|^t w^{pt/q^s} dx \right)^{qs/pt} \chi_R(x) \right) d\mu(x) \\
 (3.13) \quad & = C_{10} \sum_{R \in \mathcal{V}} \left(\int_{\sigma R} |v - c|^t w^{pt/q^s} dx \right)^{qs/pt} \\
 & \leq C_{11} \left(\int_{\Omega} |v - c|^t w^{pt/q^s} dx \right)^{qs/pt}
 \end{aligned}$$

by the condition i). Combining (3.8), (3.9) and (3.13), we have proved the theorem for the case $1 \leq s < \infty$; thus, we have completed the proof of Theorem 3.4. \square

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DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306-3027

Current address: Department of Mathematics and Statistics, University of Minnesota, Duluth, Minnesota 55812-2496

E-mail address: sding@d.umn.edu