TOPOLOGICAL ENTROPY FOR GEODESIC FLOWS
UNDER A RICCI CURVATURE CONDITION

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Abstract. It is known that the topological entropy for the geodesic flow on a
Riemannian manifold $M$ is bounded if the absolute value of sectional curvature $|K_M|$ is bounded. We replace this condition by the condition of Ricci curvature
and injectivity radius.

1. Introduction

The topological entropy is the most important invariant related to the orbit
growth of a dynamical system. It represents the exponential growth rate for the
number of orbit segments. In a sense, topological entropy describes the total expo-
nential complexity of the orbit structure.

The topological entropy for geodesic flows is closely related to the curvatures
of manifolds since geodesic flows depend on the metrics of manifolds. Let $\phi_t$ be a
geodesic flow on a Riemannian manifold $M$ and $h(\phi_t)$ be the topological entropy
for $\phi_t$. It was known that if the absolute value of sectional curvature satisfies
$|K_M| < k$, then the topological entropy for geodesic flows satisfies $h(\phi_t) \leq (n-1)\sqrt{k}$
by [Ma2]. Manning also proved that in the case of $K_M < 0$, the topological entropy
for geodesic flows is the same as the volume growth rate, $\lim_{r \to \infty} r^{-1} \log V(x, r)$, where $V(x, r)$ is the volume of the ball $B(x, r)$ contained in the universal covering
space [Ma1]. This result was extended to the case of manifolds without conjugate
points by Mañé [FM].

Bishop’s comparison theorem [GHL] implies that if Ricci curvature satisfies
$\text{Ric}_M \geq -k$ and the injectivity radius of the universal covering space of $M$ is
infinite, then $h(\phi_t) \leq \sqrt{(n-1)k}$.

We prove the following theorem;

Theorem. Let $k, i_0$ be positive real numbers. Then there exists a constant $C(i_0, n, k)$
such that for every $n$-dimensional compact Riemannian manifold $M$ with $\text{Ric}_M \geq
-k$, $\text{inj}_M \geq i_0$, the topological entropy for geodesic flow of $M$ is bounded by
$C(i_0, n, k)$, where $\text{inj}_M$ is the injectivity radius of $M$.

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For the proof, we will use Brocks’ estimate for the Laplacian of the distance function, which will play an important role on estimating the norm of Jacobi fields.

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2. Preliminaries

Let $X$ be a compact metric space with the distance function $d$, and let $\phi = \{\phi_t : X \to X\}$ be a flow, i.e. 1-parameter subgroup of homeomorphisms.

Define
\[ d^\phi_T(x, y) := \max\{d(\phi_t(x), \phi_t(y)) | 0 \leq t \leq T\}. \]

Let $S_d(\phi, \epsilon, T)$ be the minimal number of balls of radius $\epsilon$ in the metric $d^\phi_T$ which cover $X$. Define
\[ h_d(\phi, \epsilon) := \limsup_{T \to \infty} \frac{1}{T} \log S_d(\phi, \epsilon, T). \]

Then the *topological entropy* for $\phi$ is defined as follows:
\[ h(\phi) := \lim_{\epsilon \to 0} h_d(\phi, \epsilon). \]

The following proposition is Brocks’ estimate on the Laplacian of the distance function.

**Proposition ([B, DSW]).** Let $M$ be an $n$-dimensional complete Riemannian manifold with $\text{Ric}_M \geq -k$, $\text{inj}_M \geq i_0$. Let $r$ be a distance function from $p$ and $\gamma$ be a geodesic from $p$. Consider $r$ and $\Delta r$ as functions of $t$ on $\gamma$. Then $C_1(i_0, n, k) \leq \Delta r - \frac{n-1}{k} \leq C_2(i_0, n, k)$ for some constants $C_1, C_2$ depending only on $i_0, n, k$ on $[0, i_0/2]$.

The following lemma is important to estimate the upper bound of the topological entropy. It is proved in the discrete time case in [KH]. In this case, almost the same proof can be applied.

**Lemma 1.** Let $X$ be a compact metric space and $\phi = \{\phi_t\}$ be a flow on $M$ with $d(\phi_t(x), \phi_t(y)) \leq e^{Ct}d(x, y)$ for some constant $C$. Then $h(\phi) \leq |C|D(X)$ where $D(X)$ is the ball dimension defined by $D(X) := \lim_{\epsilon \to 0} \log b(\epsilon)$, and $b(\epsilon)$ is the minimum number of a covering of $X$ by $\epsilon$-balls.

**Proof.** By assumption, $d(\phi_t(x), \phi_t(y)) \leq e^{Ct}d(x, y)$, for $t \leq T$. Then we easily know $\phi_t(B(x, \epsilon/e^{Ct})) \subset B(\phi_t(x), \epsilon)$, for $t \leq T$. So $B_d(x, \epsilon/e^{Ct}) \subset B_{d_T}(x, \epsilon)$ and $S(\phi, \epsilon, t) \leq b(\epsilon/e^{Ct})$, where $B_d(x, r)$ is the $r$-ball in $d$-metric.

Then we obtain
\[
\limsup_{T \to \infty} \frac{1}{T} \log S(\phi, \epsilon, T) \leq \limsup_{T \to \infty} \frac{1}{T} \log b(\epsilon/e^{Ct}) \\
\leq \limsup_{T \to \infty} \frac{1}{T} \log(\epsilon/e^{Ct}) \leq D(X)|C|.
\]

It is known that $D(X) = \dim X$, when $X$ is a topological manifold.
3. Proof of Theorem

In the calculation of the entropy, it is an important step to define a metric on the unit tangent bundle, $T_1 M$. Let $\gamma_0(t)$ be the geodesic on $M$ with $\gamma_0'(0) = v_p$.

In Manning’s paper [Ma1], a metric on $T_1 M$ is defined as follows:

$$d_1(v_p, w_q) := \sup_{0 \leq t \leq 1} d(\gamma_0(t), \gamma_w(t)).$$

In [KH], they used the following metric:

$$d_2(v_p, w_q) := d(p, q) + \|P^q_p(v_p) - w_q\|,$$

where $P^q_p(v_p)$ is the parallel translation of $v_p$ along the geodesic from $p$ to $q$.

Now we construct a metric on $T_1 M$ as follows. Since $\text{inj}_M \geq i_0$, it is possible to identify $v_p \in T_1 M$ with the geodesic $\gamma_0(t), t \in [0, i_0/4]$ with $\gamma_0'(0) = v_p$. Then we may consider a Jacobi field $J$ along $\gamma_0(t)$ as a tangent vector at $v_p \in T_1 M$, where $J$ is generated by a $C^\infty$-rectangle $Q(t, s)$ such that $\frac{\partial Q}{\partial s}(t, 0) = J(t), \frac{\partial Q}{\partial t}(t, s)\| = 1$ for any $s$ and $0 \leq t \leq i_0/4$ and $Q(t, s_0)$ is a geodesic for any fixed $s_0$. In this identification, we consider a $C^\infty$-rectangle, $Q(t, s)$, where $0 \leq s \leq 1$ and $0 \leq t \leq i_0/4$, as a $C^\infty$-curve on $T_1 M$ from $\frac{\partial Q}{\partial t}(0, 0)$ to $\frac{\partial Q}{\partial t}(0, 1)$.

Then we define an inner product of tangent vectors and a distance on $T_1 M$ as follows.

**Definition.** Let $J_1, J_2$ be tangent vectors on $T_1 M$, i.e. Jacobi fields with above property. Then

$$(J_1, J_2) = \int_0^{i_0/4} \langle J_1(t), J_2(t) \rangle dt,$$

$$d(v_p, w_q) = \inf \int_0^1 \left( \frac{\partial Q}{\partial s}, \frac{\partial Q}{\partial t} \right)^2 ds = \inf \int_0^1 \|\frac{\partial Q}{\partial s}\| ds,$$

where the inf is taken over piecewise $C^\infty$-curves $Q$ on $T_1 M$ from $v_p$ to $w_q$, i.e. $\frac{\partial Q}{\partial t}(0, 0) = v_p$ and $\frac{\partial Q}{\partial t}(0, 1) = w_q$.

Since $T_1 M$ is a compact Hausdorff space, the above metrics induce the same topology. Now we will prove a key lemma.

**Lemma 2.** Let $M$ be a complete Riemannian manifold with $\text{Ric}_M \geq -k$, $\text{inj}_M \geq i_0$ and let $\gamma(t)$ be a minimal geodesic starting from $q$ and $J(t)$ is a Jacobi field along $\gamma$ such that $J(0) = 0$ and $\langle J'(0), \gamma'(0) \rangle = 0$. Then $\|J\|(t) \leq e^{D t} \|J\|(0)$ for some constant $D(i_0, n, k)$ if $t < i_0/2$.

**Proof.** The first half of the proof of this lemma is contained in the proof of the proposition 5.1 of [DSW]. Let $v, w \in T_q M$ and $\|v\| = \|w\| = i_0 \leq i_0/2$. Define $Q(t, s) := \exp(tV(s))$, where $V(s)$ is the geodesic on $S^{n-1}$ such that $V(0) = v$, $V(s_0) = w$. Let $\gamma(t) = \exp_q tV(0)$ and $r(x) = d(q, x)$. Then $J(t) = \frac{\partial Q}{\partial s}(t, 0)$ is a Jacobi field with above property. Then,

$$J' = \nabla_{\gamma'} J = \nabla_J \gamma' = \nabla_J \nabla r = \nabla \nabla r(J).$$

Define $A := \nabla \nabla r = \text{Hess } r$, so $\text{tr} A = \Delta r$ and $J' = AJ$.
Now we will estimate the \(|A|\). Write \(A(t) = B(t) + I/t\). From the Jacobi equation \(J'' + R(J, T)T = 0\) and \(J' = AJ\), we obtain the Riccati equation \(A' + A^2 + R = 0\) and substituting \(A(t) = B(t) + I/t\), we get \(B' + B^2 + 2/t B + R = 0\).

Then, \(\text{tr} B' + ||B||^2 + 2\text{tr} B + \text{Ric}(\gamma') \leq 0\) since \(||B||^2 \leq \text{tr} B' B = \text{tr} B^2\), where \(B'\) is the transpose of \(B\) and \(B\) is a symmetric matrix.

Multiplying the above equality by a factor \(t^{\frac{1}{2}}\) and integrating along \(\gamma(t)\), we obtain the following inequality:

\[
\int_0^{t_0} ||B||^2 t^{\frac{1}{2}} \leq -I_0^2 \text{tr} B(t_0) - \frac{3}{2} \int_0^{t_0} t^{-\frac{1}{2}} \text{tr} B(t) - \int_0^{t_0} t^{\frac{1}{2}} \text{Ric}(\gamma').
\]

Also using the Proposition ([B, DSW]) in §2, \(|\Delta r - \frac{n-1}{t}| \leq C(i_0, n, k)\) for some constant \(C(i_0, n, k)\) on \([0, i_0/2]\). Then we get \(\text{tr} B = |\text{tr} A - \frac{n-1}{t}| = |\Delta r - \frac{n-1}{t}| \leq C(i_0, n, k)\). So \(\int_0^{t_0} ||B||^2 t^{\frac{1}{2}} \leq D_1(i_0, n, k)\), for some constant \(D_1\).

Using the Hölder inequality,

\[
\int_0^{t_0} ||B|| \leq (\int_0^{t_0} t^{-\frac{1}{2}})^{\frac{1}{2}} (\int_0^{t_0} t^{\frac{1}{2}} ||B||^2)^{\frac{1}{2}} \leq D(i_0, n, k).
\]

Then we have

\[
||J'|| \leq ||J'|| + ||(B + \frac{I}{t})(J)|| \leq ||B|| ||J|| + \frac{||J||}{t},
\]

\[
\int_0^{t_0} \frac{||J'||}{||J||} \leq D + \log l_0 - \log \delta,
\]

\[
\log(\frac{||J||}{||J||_0}) \leq D + \log(l_0/\delta),
\]

\[
||J||_0 \leq e^{D_1} \frac{||J||_0}{\delta},
\]

\[
||J||_0 \leq \lim_{l_0 \to 0} e^{D_1} \frac{||J||_0}{\delta} = e^{D_1} ||J'||(0),
\]

which completes the proof.

From Lemma 5.2 in [DSW], we know that \(||J(t)|| \leq \frac{t}{t_0} e^D ||J(t_0)||\), \(0 \leq t \leq t_0 < i_0\). Then \(||J'(0)|| = \lim_{t \to 0} \frac{||J(t)||}{t} \leq \frac{e^D ||J(t_0)||}{t_0} \).

From lemma 2 and the above inequality, we obtain the following inequality:

(1) \(e^{-D} ||J'(0)|| t \leq ||J(t)|| \leq e^D ||J'(0)|| t\).

Also we know that

(2) \(K_1 ||J(i_0/4)|| \leq ||J'(0)|| \leq K_2 ||J(i_0/4)||\),

for some constants \(K_1, K_2\) depending only on \(i_0, n, k\). Consequently,

(3) \(K_3 ||J(i_0/4)|| t \leq ||J(t)|| \leq K_4 ||J(i_0/4)|| t\).

If \((J'(0), \gamma'(0)) \neq 0\), we can decompose \(J\) into tangential and perpendicular components and obtain the above boundedness (3), since tangential component is linear in \(t\) [DSW].
Lemma 3. If $h$ is sufficiently small,
\[ d(\phi_h(v_p), \phi_h(w_q)) - d(v_p, w_q) \leq N(i_0, n, k)d(v_p, w_q)h, \]
for some constant $N(i_0, n, k)$.

Proof. Let $Q(t, s)$ be a length minimizing curve from $v_p$ to $w_q$, i.e. a piecewise $C^\infty$-rectangle which realizes the distance from $v_p$ to $w_q$. We may assume the existence of a length minimizing curve.

Then we know that
\[ d(v_p, w_q) = \int_0^1 \int_0^{i_0/4} \| \frac{\partial Q}{\partial s} \| dsdt \]
and
\[ d(\phi_h(v_p), \phi_h(w_q)) \leq \int_0^1 \int_0^{i_0/4+h} \| \frac{\partial Q}{\partial s} \| dsdt. \]
So we get
\[ d(\phi_h(v_p), \phi_h(w_q)) - d(v_p, w_q) \leq \int_0^1 \int_0^{i_0/4} \int_0^{i_0/4+h} \int_0^1 \| \frac{\partial Q}{\partial s} \| dsdt. \]

Let $\frac{\partial Q}{\partial s}(0, s) = V(s)$ and $\frac{\partial Q}{\partial s}(i_0/4, s) = W(s)$. Decompose $J$ into $J_1$ and $J_2$ such that $J_1(0) = 0$, $J_2(i_0/4) = 0$. Then $\|J_2(0)\| = ||V||$ and $\|J_1(i_0/4)\| = ||W||$. If $||V|| = ||W|| = 0$, then $J(t) = J_1(t) = J_2(t) = 0$ for all $t \leq i_0/4$ and $v_p = w_q$. So we do not need consider this case. From now on we may assume $||V|| \neq 0$. Then by (3), we have
\[ K_3||W||t \leq ||J_1(t)|| \leq K_4||W||t, \]
(4)
\[ K_5||V||t \leq ||J_2(i_0/4 - t)|| \leq K_4||V||t. \]

Thus we have
\[ d(\phi_h(v_p), \phi_h(w_q)) - d(v_p, w_q) \leq \int_0^h \int_0^{i_0/4} \int_0^{i_0/4+h} \int_0^1 \| J(t) \| dsdt \]
\[ \leq \int_0^h \int_0^{i_0/4} \int_0^{i_0/4+h} \int_0^1 \| J_1(t) \| dsdt \]
\[ + \int_0^h \int_0^{i_0/4} \int_0^{i_0/4+h} \int_0^1 \| J_2(t) \| dsdt \]
\[ \leq (\int_0^{i_0/4+h} + \int_0^h K_4tdt)(\int_0^1 ||V|| + ||W||ds) \]
\[ \leq K(i_0, n, k)(\int_0^1 ||V|| + ||W||ds)h, \]
for some constant $K(i_0, n, k)$. Now compute the $d(v_p, w_q)$. Let
\[ a = \frac{i_0K_3||V||}{4(K_3||V|| + K_4||W||)}, \quad b = \frac{i_0K_3||W||}{4(K_3||W|| + K_4||V||)}. \]
On \([0, a]\), we know that \(||J_2(t)|| \geq ||J_1(t)||\) and on \([i_0/4 - b, i_0/4]\), \(||J_1(t)|| \geq ||J_2(t)||\) by (4) and (5). So we obtain

\[
d(v_p, w_q) = \int_0^{i_0/4} \int_0^1 ||J|| ds dt
\]

\[
\geq \int_0^1 \int_0^a ||J_2|| - ||J_1|| ds dt + \int_0^1 \int_{i_0/4 - b}^{i_0/4} ||J_1|| - ||J_2|| ds dt
\]

\[
\geq \int_0^1 \int_0^a K_3 ||V||(i_0/4 - t) - K_4 ||W|| t ds dt
\]

\[
+ \int_0^1 \int_{i_0/4 - b}^{i_0/4} K_3 ||W|| t - K_4 ||V|| (i_0/4 - t) ds dt.
\]

The integral \(\int_0^a K_3 ||V||(i_0/4 - t) - K_4 ||W|| t dt\) is the area of

\[\{(x, y) \mid K_4 ||W|| x \leq y \leq K_3 ||V||(i_0/4 - x), \ x \leq a\}\].

Hence we get

\[
\int_0^a ||J_2|| - ||J_1|| \geq a i_0 K_3 ||V|| / 8 = i_0^2 K_3 / 32 \frac{K_3 ||V||^2}{K_4 ||V|| + K_3 ||W||}.
\]

Similarly we get

\[
\int_{i_0/4 - b}^{i_0/4} ||J_1|| - ||J_2|| \geq i_0^2 K_3 / 32 \frac{K_3 ||W||^2}{K_4 ||V|| + K_3 ||W||}.
\]

It is an easy calculation that

\[
\int_0^{i_0/4} ||J|| dt \geq \frac{a_1 (||V||^2 ||W|| + ||V|| ||W||^2) + a_2 (||V||^3 + ||W||^3)}{b_1 (||W||^2 + ||V||^2) + b_2 ||V|| ||W||},
\]

for some positive constant \(a_i, b_i\) depending only on \(i_0, n, k\).

Then we obtain

\[
\frac{||V|| + ||W||}{\int_0^{i_0/4} ||J|| dt} \leq \frac{a_1 (||V||^2 ||W|| + ||V|| ||W||^2) + a_2 (||V||^3 + ||W||^3)}{b_1 (||W||^2 + ||V||^2) + b_2 ||V|| ||W||}
\]

\[
= \frac{b_1 (||V||^3 + ||W||^3) + c_1 (||V||^2 ||W|| + ||V|| ||W||^2)}{a_2 (||V||^3 + ||W||^3) + d_1 (||V|| ||W||^2 + ||V|| ||W||^2)}
\]

\[
= \frac{b_1 ||V||^3 + c_1 ||V||^2 ||W|| + d_1 ||V|| ||W||^2 + b_2 ||W||^3}{a_2 ||V||^3 + c_1 ||V||^2 ||W|| + a_1 ||V|| ||W||^2 + c_2 ||W||^3}
\]

\[
= \frac{b_1 + c_1 ||W|| ||V|| + c_1 ||V||^2 ||W|| + b_2 ||W||^3}{a_2 + a_1 ||W|| ||V|| + a_1 ||V||^2 ||W|| + a_2 ||W||^3}
\]

\[
\leq C_0(i_0, n, k),
\]

for some positive constants \(c_1(i_0, n, k), C_0(i_0, n, k)\), since we assume \(||V|| \neq 0\) and we know the boundedness of \(\frac{e_0 + e_1 x + e_2 x^2 + e_3 x^3}{f_0 + f_1 x + f_2 x^2 + f_3 x^3}\) from

\[
\lim_{x \to 0} \frac{e_0 + e_1 x + e_2 x^2 + e_3 x^3}{f_0 + f_1 x + f_2 x^2 + f_3 x^3} = \frac{e_0}{f_0}.
\]
and
\[
\lim_{x \to \infty} \frac{e_0 + e_1x + e_2x^2 + e_3x^3}{f_0 + f_1x + f_2x^2 + f_3x^3} = \frac{e_3}{f_3}
\]
for positive constants \(e_i, f_i\) and \(x \geq 0\).

Consequently, we have
\[
e_0 + e_1x + e_2x^2 + e_3x^3 = e_3 f_3
\]
for positive constants \(e_i, f_i\) and \(x \geq 0\).

References


