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ISOTOPY AND IDENTITIES IN ALTERNATIVE ALGEBRAS

M. BABIKOV

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Abstract. In this paper we show how to construct an isomorphism between an alternative algebra A over a field of characteristic $\neq 3$ and its isotope $A^{(1+c)}$, where c is an element of Zhevlakov's radical of A. This leads to the equivalence of any polynomial identity $f = 0$ in alternative algebras and the isotope identity $f^{(s)} = 0$.

Given an invertible element s of an alternative algebra A, we can form a new algebra by taking the same linear structure but a new multiplication

$$x *_s y = (xs)y.$$

The resulting algebra, denoted $A^{(s)}$, is also alternative (see [1]) and is called an s-isotope. Associative and Cayley isotopes are always isomorphic [2], and an isotope of a finite-dimensional alternative algebra over an algebraically closed field of characteristic $\neq 3$ is isomorphic to the original algebra as well [1]. We consider the case of an arbitrary alternative algebra over a field of characteristic $\neq 3$ with nonzero Zhevlakov radical, and a particular choice of s, and construct an explicit isomorphism between A and $A^{(s)}$. As a consequence, we derive the equivalence of any polynomial identity and its isotope in an arbitrary alternative algebras over a field of characteristic $\neq 3$. The paper has benefited from many discussions with J. Ferrar, and I would like to thank him for his valuable help and encouragement.

Theorem 1. *There is a set of coefficients $t^i_j, i, j \geq 0$, such that $t^0_0 = 1$ and for any integer m, any alternative algebra A over a field of characteristic $\neq 3$ and any elements a, b, c from A such that $\mathrm{Id}_A(c)^m = 0$, the polynomial $t(x) = \sum_{i,j} t^i_j c^i x c^j$ satisfies*

$$t(a) *_{1+c} t(b) = t(ab). \tag{1}$$

Proof. We will show the way to calculate the coefficients t^i_j such that (1) holds for any alternative algebra A. Clearly,

$$\begin{aligned} t(a) *_{1+c} t(b) &= t(a)t(b) + (t(a)c)t(b) \\ &= \sum t^i_j t^k_l (c^i a c^j)(c^k b c^l) + \sum t^i_j t^k_l (c^i a c^{j+1})(c^k b c^l) \\ &= \sum t^k_l (t^i_j + t^i_{j-1})(c^i a c^j)(c^k b c^l), \end{aligned}$$

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where $t^i_{-1} = 0$ and $0 \le i, j, k, l \le m-1$. For brevity we shall denote $t^k_l(t^i_j + t^i_{j-1})$ simply by t^{ki}_{lj}; then

$$\begin{aligned} t(a) *_{1+c} t(b) - t(ab) &= \sum t^{ki}_{lj}((c^i a c^{j+k})(bc^l) + (c^i a c^j, c^k, bc^l)) \\ &\quad - \sum t^i_l((c^i a)(bc^l) - (c^i, a, b)c^l + (c^i a, b, c^l)) \\ &= \sum_{i,l,n} \sum_{j+k=n} t^{ki}_{lj}(c^i a c^n)(bc^l) + \sum t^{ki}_{lj} c^{j+l}(a, b, c^k)c^i \\ &\quad - \sum t^i_l(c^i a)(bc^l) - \sum t^i_l(c^i a, b, c^l) + \sum t^i_l(a, b, c^i)c^l. \end{aligned} \tag{2}$$

We need (1) to be true for any alternative algebra, in particular for an associative one, for which all the associators in (2) are zeroes and (1) is equivalent to

$$\sum_{i,l,n} \sum_{j+k=n} t^{ki}_{lj} c^i a c^n b c^l - \sum t^i_l c^i a b c^l = 0. \tag{3}$$

For the cases $n = 0$ and $n \ge 1$ we get that

$$t^{0i}_{l0} = t^i_l, \tag{4}$$

$$\sum_{j+k=n} t^{ki}_{lj} = 0, \tag{5}$$

for any i and l. It is easy to see that (4) yields

$$t^i_l = t^i_0 \cdot t^0_l. \tag{6}$$

But then

$$\sum_{j+k=n} t^{ki}_{lj} = t^i_l \sum_{j+k=n} (t^k_j + t^k_{j-1}),$$

and therefore (5) yields

$$\sum_{j+k=n} t^k_j = - \sum_{j+k=n-1} t^k_j.$$

Consequently, since $\sum_{j+k=1} t^k_j = t^0_0 = 1$ and by induction on n,

$$\sum_{j+k=n} t^k_j = (-1)^n. \tag{7}$$

Now we assume (6) and (7), which implies (3). Going back to the alternative case we note that (4) and (5) also imply

$$\sum_{i,l,n} \sum_{j+k=n} t^{ki}_{lj}(c^i a c^n)(bc^l) - \sum t^i_l(c^i a)(bc^l) = 0.$$

So in the expression (2) only sums with the associators are left. It is easy to see that in any alternative algebra the Moufang identities imply

$$\begin{aligned} (a, b, c^{k+l}) &= -(a, c^l, c^k b) + b(a, c^l, c^k) + c^l(a, b, c^k) \\ &= (a, b, c^l)c^k + c^l(a, b, c^k). \end{aligned}$$

Using this identity we get

$$\begin{aligned} t(a) * t(b) - t(ab) &= \sum t_{lj}^{ki}(-(a,b,c^{l+j})c^{k+i} + (a,b,c^{l+j+k})c^i) \\ &\quad - \sum t_l^i(a,b,c^l)c^i + \sum t_l^i(a,b,c^i)c^l \\ &= \sum T_{pq}(a,b,c^p)c^q, \end{aligned} \tag{8}$$

where

$$T_{pq} = t_q^p - t_p^q - \sum_{l+j=p,k+i=q} t_{lj}^{ki} + \sum_{l+j+k=p,(i=q)} t_{lj}^{kq}.$$

Let us simplify this expression for T_{pq}. Clearly by (7)

$$\begin{aligned} \sum_{l+j+k=p} t_{lj}^{kq} &= \sum_{k+l=p-j} t_l^k \sum_j t_{0g}^{0j} = \sum_j (-1)^{p-j} t_{0g}^{0j} \\ &= \sum_j (-1)^{p-j}(t_j^q + t_{j-1}^q) = t_p^q, \end{aligned}$$

and so

$$T_{pq} = t_q^p - \sum_{l+j=p,k+i=q} t_{lj}^{ki}.$$

Now we use induction on $d = p + q$ to prove the existence of t_0^d and t_d^0 such that $T_{pq} = 0$ for any $p + q \leq d$. Note that although we do it for any p and q, we need $T_{pq} = 0$ only for $p \neq 0$, because otherwise in (8) we have $(a, b, c^p)c^q = 0$. First, we consider the case $q = 0$ and find t_0^d:

$$t_0^d + t_d^0 = \sum_{i+j=d} t_j^i - \sum_{i+j=d,i\neq d,j\neq d} t_j^i = (-1)^d - \sum_{i+j=d,i\neq d,j\neq d} t_j^0 t_0^i.$$

So, since $T_{pq} = 0$,

$$\begin{aligned} t_0^d &= \sum_{l+j=d} t_l^0(t_j^0 + t_{j-1}^0) = \sum_{l+j=d,l\neq d,j\neq d} t_l^0(t_j^0 + t_{j-1}^0) + t_{d-1}^0 + 2t_d^0 \\ &= \sum_{l+j=d,l\neq d,j\neq d} t_{lj}^{00} + t_{d-1}^0 2\left((-1)^d - \sum_{i+j=d,i\neq d,j\neq d} t_j^0 t_0^i - t_0^d\right), \end{aligned} \tag{9}$$

and we get

$$t_0^d = \frac{1}{3}\left(\sum_{l+j=d,l\neq d,j\neq d} t_{lj}^{00} + t_{d-1}^0 + 2\left((-1)^d - \sum_{i+j=d,i\neq d,j\neq d} t_j t^i\right)\right).$$

To complete the proof we must show that $T_{qp} = 0$ for any p, q such that $p + q = d$ based on the assumption that this is true for $p + q \leq d - 1$. If $p \neq 0$ and $q \neq 0$, then

$$T_{pq} = t_q^p - \sum_{k+i=q} t_{00}^{ki} \sum_{l+j=p} t_{lj}^{00} = t_q^p - t_0^p t_q^0 = 0.$$

The case $q = 0$ follows from (9) and for $p = 0$ we have

$$\begin{aligned} T_{0d} &= t_d^0 - \sum_{k+i=d} t_0^k t_0^i = t_d^0 - \sum_{k+i=d} \left(\sum_{l+j=k} t_{lj}^{00} \right) t_0^i \\ &= t_d^0 - \sum_{j \le d} t_{0j}^{00} \sum_{i+l=d-j} t_l^i = t_d^0 - \sum_{j \le d} (t_j^0 + t_{j-1}^0)(-1)^{d-j} \\ &= t_d^0 - \left(\sum_{j \le d} t_j^0 (-1)^{d-j} - \sum_{j<d} (t_j^0)(-1)^{d-j} \right) = 0. \end{aligned}$$

This proves the theorem. □

Corollary 1. *Let A, c and $t(x)$ be the same as in the Theorem 1; then the mapping $t : x \to t(x)$ is an isomorphism of A onto $A^{(1+c)}$.*

Proof. Theorem 1 states that t is a homomorphism. To prove that t is surjective, we have to find a polynomial $T = t^{-1}$ such that $t(T(a)) = a$. Let us use induction on $m = \deg(c)$. For $m = 1$ the statement is obvious: $T = 1$. Assume that T_1 is the required polynomial for the case of $\deg(c) = m - 1$; then for $\deg(c) = m$ we have $t(T_1(a)) = a + R(a)$, where R is a homogeneous polynomial linear in a and of degree $m - 1$ in c. We set

$$T(a) = T_1(a) - T_1(R(a)).$$

Then $R(R(a)) = 0$, since $R(R(a))$ is of degree $2(m - 1) \ge m$ in c, and we have

$$t(T(a)) = a + R(a) - R(a) - R(R(a)) = a - R(R(a)) = a.$$

This completes the proof. □

Let $x_1, ..., x_k$ be some generators of a free alternative algebra A_0, and f a polynomial in $x_1, ..., x_k$. Since A_0 is free, for any $s \in A_0$ there is a homomorphism $\phi_s : A_0 \to A_0^{(s)}$ such that $x_i \mapsto x_i$. We denote the image of f by $f^{(s)}$.

Theorem 2. *Any alternative algebra A over a field of characteristics $\neq 3$ with a polynomial identity $f = 0$ satisfies also $f^{(s)} = 0$ for any $s \in A$.*

Proof. The variety of alternative algebras is homogeneous ([3], page 8); therefore we need to consider only the case of homogeneous polynomial f. Let A_0 be a free alternative algebra on $k+1$ generators $x_1, ..., x_k, c$. By Theorem 1 and the corollary,

$$f^{(1+c)}(x_1, ..., x_k) = t(f(t^{-1}x_1, ..., t^{-1}x_k)) + c_m,$$

where $c_m \in \mathrm{Id}(c)^m$ and $m = \deg(f)$. Consider the homogeneous component of degree $m - 1$ in c:

$$f^{(c)}(x_1, ..., x_k) = \Delta_c^{m-1} t(f(t^{-1}x_1, ..., t^{-1}x_k)) \in \mathrm{Id}(f). \tag{10}$$

Here Δ_c^{m-1} is the linearization operator (see [3]), and $\mathrm{Id}(f)$ denotes the ideal generated by all values of f. Since A_0 is free and $x_1, ..., x_k, c$ are its generators, (10) holds for any algebra A and for any elements $x_1, ..., x_k, c$ in A. This proves the theorem. □

For algebras over a field of characteristics 3 Theorem 2 is false. Consider a commutative but not associative alternative algebra; it satisfies $[x, y] = 0$. On the other hand,

$$[x, y]^{(c)} = (xc)y - (yc)x = (x, c, y).$$

So, generally speaking, $[x, y]^{(c)} \neq 0$.

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Department of Mathematics, Ohio State University, Columbus, Ohio 43202
E-mail address: brkvch@math.ohio-state.edu