NON-EXISTENCE AND UNIQUENESS RESULTS
FOR BOUNDARY VALUE PROBLEMS
FOR YANG-MILLS CONNECTIONS

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Abstract. We show uniqueness results for the Dirichlet problem for Yang-Mills connections defined in \( n \)-dimensional (\( n \geq 4 \)) star-shaped domains with flat boundary values. This result also shows the non-existence result for the Dirichlet problem in dimension 4, since in 4-dimension, there exist countably many connected components of connections with prescribed Dirichlet boundary value. We also show non-existence results for the Neumann problem. Examples of non-minimal Yang-Mills connections for the Dirichlet and the Neumann problems are also given.

1. Introduction

Let \( M \) be a Riemannian manifold with boundary and \( G \) a compact Lie group. Let \( A_0 \) be a given smooth connection defined on a principal \( G \)-bundle \( P_0 \rightarrow \partial M \). We denote by \( \mathcal{A}(A_0) \) the space of smooth connections defined in principal \( G \)-bundles over \( M \) with Dirichlet boundary value \( A_0 \) at \( \partial M \). That is,

\[
\mathcal{A}(A_0) = \{ A : A \text{ is a smooth connection defined in some principal } G\text{-bundle over } M \text{ with } i^* A \sim A_0 \text{ over } \partial M \},
\]

where \( i^* A \sim A_0 \) means that \( i^* A \) is gauge equivalent to \( A_0 \) over \( \partial M \) and \( i : \partial M \rightarrow M \) is the inclusion map.

By definition, a connection \( A \) is a solution to the Dirichlet problem for Yang-Mills equations defined in \( M \) with boundary value \( A_0 \) at \( \partial M \) if \( A \in \mathcal{A}(A_0) \) with finite energy and \( A \) is Yang-Mills, that is, \( D_A^* F_A = 0 \) in \( M \). Here \( D_A^* \) is the formal adjoint of the covariant exterior derivative \( D_A = d + [A, \cdot] \) with respect to the \( L^2 \)-metric on \( \Lambda^2 T^* M \otimes \text{Ad}(P) \) induced from the Riemannian metric on \( M \) and adjoint invariant metric on \( g \), the Lie algebra of \( G \).

We also recall the definition of the Neumann problem. A connection \( A \) is a solution to the Neumann problem for Yang-Mills equations if and only if \( A \) is Yang-Mills in \( M \) with finite energy and \( i^* (\ast F_A) = 0 \) on \( \partial M \). Here \( \ast : \Lambda^2 T^* M \otimes \text{Ad}(P) \rightarrow \Lambda^2 T^* M \otimes \text{Ad}(P) \) is the Hodge star operator.

Dirichlet and Neumann problems for Yang-Mills connections were first defined and studied by Marini \cite{7}. In \cite{7}, Marini showed the existence and regularity of

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absolute minimum solutions for the Dirichlet problem, and Neumann problem with prescribed class $\eta \in H^2(M; \pi_1(G))$, when $\dim M = 4$.

In [4], Isobe and Marini studied the existence of “topologically distinct solutions” to the Dirichlet problem when $M = B^4 = \{x \in \mathbb{R}^4 : |x| \leq 1\}$ and $G = SU(2)$ using the fact that in this case $A(A_0)$ is the disjoint union of infinitely many connected components $A_k$ indexed by $k \in \mathbb{Z}$ (specifically, for some fixed $B \in A(A_0)$, $A_k = \{A : C(B) - C(A) = k\}$, where $C(A) = \frac{1}{8\pi^2} \int_M \text{tr}(F_A \wedge F_A) \wedge 1$). It is shown that for “generic boundary values”, there exist infinitely many topologically distinct solutions, and for any non-flat boundary values, there exist at least two topologically distinct solutions. These solutions all minimize the action on their component. In [4] we also showed that for some boundary values $A_0$ the action attains its infimum on only finitely many components, and hence does not attain its infimum on infinitely many others.

But the following question remained for the Dirichlet problem: Is there a Yang-Mills connection (not necessarily minimizing) in each component of $A(A_0)$?

In this paper, we show that for a flat boundary value $A_0$, a flat connection is the only solution for the Dirichlet problem with boundary value $A_0$ when the base manifold is star-shaped (see §2 for the definition) and structure group $G$ is an arbitrary compact Lie group. Thus we cannot, in general, expect the existence of Yang-Mills connections in each connected component of $A(A_0)$.

We also show that this uniqueness result does not hold for general $M$. We give an example of $M$ (annulus) such that there exists a non-flat Yang-Mills connection on some principal bundle over $M$ which is flat at $\partial M$. This connection is necessarily a non-minimal Yang-Mills connection. The construction of this connection comes from the one given in [9], where Parker constructed non-minimal Yang-Mills connections over $S^4$ or $S^3 \times S^1$. However, in our case, the argument is simplified by using a direct variational method.

As for the Neumann problem, if $\eta \in H^2(M; \pi_1(G))$ is trivial, the solution obtained by Marini [7] is a flat connection. This raises the following problem: Does there exist a non-flat connection for the Neumann problem if $\eta \in H^2(M; \pi_1(G))$ vanishes?

We show in this paper, when $\dim M = 4$, for star-shaped domains, a flat connection is the only solution for the Neumann problem. We also give an example of a non-flat Yang-Mills connection on some principal $SU(2)$-bundle $P$ with $\eta(P) = 0$ which satisfies the Neumann condition.

See also [14] for a uniqueness result to the Dirichlet problem for (anti-)self-dual connections. Note that we do not restrict ourselves to (anti-)self dual connections.

Our result also shows similarities between our results and other non-existence (or uniqueness) results related to Yamabe equations ([10]), harmonic mappings ([6]), constant mean curvature equations ([17]), etc...

2. PROOF OF THE MAIN RESULTS

Our main results are the following. The first result is concerned with the Dirichlet problem:

**Theorem 2.1.** (1) Let $n \geq 5$, $M$ a $C^2$-star-shaped bounded domain in $\mathbb{R}^n$ with flat metric, and $G$ a compact Lie group. Let $A_0$ be a flat connection on some principal $G$-bundle $P_0 \to \partial M$. Assume $A \in A(A_0)$ is a solution to the Dirichlet problem
for Yang-Mills connections. Then $A$ is a flat connection, that is, the curvature $F_A = dA + A \wedge A$ of $A$ vanishes.

(2) Let $n = 4$ and $G$ be as in (1). Assume $M$ is a $C^2$-strictly star-shaped bounded domain in $\mathbb{R}^4$ with flat metric. Then the same conclusion as in (1) holds.

Our next result is concerned with the Neumann problem:

**Theorem 2.2.** Let $n = 4$ and $G$ a compact Lie group. Let $M$ be a $C^2$-strictly star-shaped bounded domain in $\mathbb{R}^4$ with flat metric. Assume $A$ is a solution to the Neumann problem for Yang-Mills connections. Then $A$ is a flat connection.

Before we give the proofs of the above theorems, we give here the definitions of star-shaped and strictly star-shaped domains.

**Definition 2.3.** (1) A domain $M \subset \mathbb{R}^n$ is called star-shaped if there exists a point $x_0 \in M$ such that the line segment $xx_0$ is contained in $M$ for all $x \in M$. In this case, we have $\langle x - x_0, \nu(x) \rangle \geq 0$ for any point $x \in \partial M$, where $\nu(x)$ is the outer normal at $x \in \partial M$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$.

(2) A domain $M \subset \mathbb{R}^n$ is called strictly star-shaped if $M$ is star-shaped and $\langle x - x_0, \nu(x) \rangle > 0$ for any $x \in \partial M$, where $x_0$ is as in (1).

Note that the star-shaped domain is contractible. Therefore for such base manifold $M$ and any compact Lie group $G$, $\eta \in H^2(M; \pi_1(G))$ is always trivial.

**Proof of the Theorem 2.1 and Theorem 2.2.** Both theorems follow from the following first variation formula for Yang-Mills fields (see [11]):

**Lemma 2.4.** Let $\{e_1, \ldots, e_n\}$ be an orthonormal tangent frame for $TM$. Let $A$ be a solution to the Dirichlet or the Neumann problem for Yang-Mills equations. Then the following holds for any vector field $X$ in $M$ with compact support in $M^\circ$:

\[
\int_M |F_A|^2 \, d\text{vol}X - 4\langle F_A(\nabla e_i X, e_j), F_A(e_i, e_j) \rangle = 0.
\]

(2.1)

Here $\langle \cdot, \cdot \rangle$ in the above equation is the adjoint invariant inner product of $g$.

In [11] this result is stated for Yang-Mills connections without boundary conditions, however, this is also true for Yang-Mills connections with boundary value conditions, since the variation used in [11] does not change boundary values.

We first prove Theorem 2.1.

Without loss of generality we may assume that $0 \in M$ and $M$ is (strictly) star-shaped with respect to the point $x_0 = 0$ (see Definition 2.3).

For $\delta > 0$, define

\[
O_\delta(\partial M) := \{x \in M : d(x, \partial M) < \delta\}.
\]

Since $\partial M$ is $C^2$, there exists $\delta > 0$ such that the map $\pi : O_\delta(\partial M) \to \mathbb{R}$ defined by $\pi(x) = d(x, \partial M)$ is $C^1$.

For such $\delta$ define the map $\Pi : M \to \mathbb{R}$ as

\[
\Pi(x) := \begin{cases} 
\pi(x) & \text{if } x \in O_\delta(\partial M), \\
\delta & \text{if } x \in M \setminus O_\delta(\partial M).
\end{cases}
\]

$\Pi$ is a Lipschitz function defined in $M$. 

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Next define the map \( \rho_\epsilon : \mathbb{R} \to \mathbb{R} \) for \( \epsilon > 0 \) as

\[
\rho_\epsilon := \begin{cases} 
0 & \text{if } x \leq \epsilon, \\
1 & \text{if } x \geq 2\epsilon, \\
\frac{x}{\epsilon} - 1 & \text{if } \epsilon \leq x \leq 2\epsilon.
\end{cases}
\]

For \( \epsilon > 0 \) with \( 2\epsilon < \delta \), define the vector field \( X_\epsilon \) in \( M \) by

\[
(2.2) \quad X_\epsilon = \rho_\epsilon(\Pi(x)) \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.
\]

Note that the vector field \( X_\epsilon \) is only Lipschitzian, and the formula (2.1) holds for Lipschitzian vector fields by a density argument. Also note that \( \text{supp}(X_\epsilon) \subset \{ x \in M : \Pi(x) \geq \epsilon \} \subset M^o \).

We insert this vector field \( X_\epsilon \) in the first variational formula (2.1), taking \( e_i = \frac{\partial}{\partial x_i} \). A short calculation gives

\[
(2.3) \quad 0 = (n - 4) \int_M \rho_\epsilon(\Pi(x))|F_A|^2 \, dx + \int_M \rho_\epsilon(\Pi(x))\langle \nabla \Pi(x), x \rangle |F_A|^2 \, dx \\
- 4 \int_M \rho_\epsilon(\Pi(x)) \left( F_A \left( x_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j} \right), F_A \left( \nabla \Pi(x), \frac{\partial}{\partial x_j} \right) \right) \, dx.
\]

Here and in the following, we use the summation convention.

Letting \( \epsilon \downarrow 0 \) in (2.3), we obtain

\[
(2.4) \quad 0 = (n - 4) \int_M |F_A|^2 \, dx - \int_{\partial M} \langle x, \nu(x) \rangle |F_A|^2 \\
+ 4 \int_{\partial M} \left( F_A \left( x_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial \tau_l} \right), F_A \left( \frac{\partial}{\partial \nu}, \frac{\partial}{\partial \tau_l} \right) \right). \tag{2.4}
\]

We rewrite (2.4) using the new tangent frame \( \{ \nu, \tau_1, \ldots, \tau_{n-1} \} \) at \( \partial M \). Here \( \{ \tau_1, \ldots, \tau_{n-1} \} \) is an orthonormal tangent frame of \( \partial M \). Then \( \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k} = \langle x, \nu \rangle \frac{\partial}{\partial \nu} \) and (2.4) becomes

\[
(2.5) \quad 0 = (n - 4) \int_M |F_A|^2 \, dx - \int_{\partial M} \langle x, \nu \rangle |F_A|^2 \\
+ 4 \int_{\partial M} \langle x, \tau_k \rangle (F_A(\tau_k, \tau_l), F_A(\nu, \tau_l)). \tag{2.5}
\]

By the Dirichlet boundary condition \( i^*A \sim 0 \), we have \( F_A \left( \frac{\partial}{\partial \tau_k}, \frac{\partial}{\partial \tau_l} \right) = 0 \) for all \( 1 \leq k, l \leq n - 1 \). Thus from (2.5), we get

\[
(2.6) \quad (n - 4) \int_M |F_A|^2 \, dx - \int_{\partial M} \langle x, \nu \rangle |F_A|^2 + 4 \int_{\partial M} \langle x, \nu \rangle i^*(F_A)^2 = 0.
\]

Using again the condition \( i^*A \sim 0 \), we reduce (2.6) to the following form:

\[
(2.7) \quad (n - 4) \int_M |F_A|^2 \, dx + 3 \int_{\partial M} \langle x, \nu \rangle |i^*(F_A)|^2 = 0.
\]

When \( n > 4 \) and \( M \) is star-shaped, (2.7) implies \( F_A = 0 \). This is the assertion of Theorem 2.1 (1).

To prove Theorem 2.1 (2), we work more.

Under the assumption of Theorem 2.1 (2) we have, by (2.7), \( i^*(F_A) = 0 \) on \( \partial M \). Combining this with the Dirichlet boundary condition, we conclude that all components of \( F_A \) vanish on \( \partial M \).
Let \( \rho > 0 \) be such that the nearest point retraction \( r: \mathcal{O}_\rho(M) \to \overline{M} \) defined by \( r(x) = x \) if \( x \in M \) and \( r(x) = y_x \) if \( x \in \mathcal{O}_\rho(M) \setminus M \), where \( y_x \in \partial M \) is the unique point satisfying \( d(x, M) = d(x, y_x) \), is well defined. Here \( \mathcal{O}_\rho(M) := \{ x \in \mathbb{R}^n : d(x, M) < \rho \}. \)

We define the connection \( \hat{A} \) defined in \( \mathcal{O}_\rho(M) \) by \( \hat{A} = r^* A \). Note that \( F_{\hat{A}} = 0 \) on \( \mathcal{O}_\rho(M) \setminus M \).

We need the following lemma.

**Lemma 2.5.** \( \hat{A} \) is a (weak) Yang-Mills connection in \( \mathcal{O}_\rho(M) \).

**Proof.** First note that

\[
F_{\hat{A}} \in L^2(\Lambda^2 T^* \mathcal{O}_\rho(M) \otimes \text{Ad}(r^* P)) \quad \text{and} \quad \hat{A} \in L^2_1(T^* \mathcal{O}_\rho(M) \otimes \text{Ad}(r^* P)).
\]

We need to show the following:

\[
\int_{\mathcal{O}_\rho(M)} (F_{\hat{A}}, D_{\hat{A}} \phi) = 0 \quad \text{for all} \quad \phi \in C^\infty(T^* \mathcal{O}_\rho(M) \otimes \text{Ad}(r^* P))
\]

with \( \text{supp}(\phi) \subset \mathcal{O}_\rho(M) \).

Fix such \( \phi \in C^\infty(T^* \mathcal{O}_\rho(M) \otimes \text{Ad}(r^* P)) \). We have

\[
(2.8) \quad \int_{\mathcal{O}_\rho(M)} (F_{\hat{A}}, D_{\hat{A}} \phi) = \int_{\mathcal{O}_\rho(M) \setminus \overline{M}} (F_{\hat{A}}, D_{\hat{A}} \phi) + \int_M (F_{\hat{A}}, D_{\hat{A}} \phi)
\]

\[
= \int_M (F_{\hat{A}}, D_{\hat{A}} \phi),
\]

since \( F_{\hat{A}} = 0 \) in \( \mathcal{O}_\rho(M) \setminus \overline{M} \). On the other hand, since \( D_{\hat{A}}^* F_{\hat{A}} = D_{\hat{A}}^* F_A = 0 \) in \( M \) and all components of \( F_{\hat{A}} \) vanish on \( \partial M \), by integration by parts, we have

\[
(2.9) \quad \int_M (F_{\hat{A}}, D_{\hat{A}} \phi) = 0.
\]

Combining (2.8) and (2.9), we complete the proof. \( \square \)

We continue the proof of Theorem 2.1 (2).

By the regularity theory for weak Yang-Mills connections over 4-manifolds [15], there exists a gauge \( g \in L^2_2(\mathcal{O}_\rho(M); G) \) such that \( \hat{A} := g^* A \in C^\infty \). Since \( \hat{A} \) is a Yang-Mills connection in \( \mathcal{O}_\rho(M) \), and by the Bianchi identity, it satisfies

\( (D_{\hat{A}} D_{\hat{A}}^* + D_{\hat{A}}^* D_{\hat{A}}) F_{\hat{A}} = 0 \).

On the other hand, since \( F_{\hat{A}} = 0 \) in \( \mathcal{O}_\rho(M) \setminus \overline{M} \), by the unique continuation theorem applied to the 2nd order elliptic partial differential operator \( D_{\hat{A}} D_{\hat{A}}^* + D_{\hat{A}}^* D_{\hat{A}} \) [1], we have \( F_{\hat{A}} = 0 \) in \( \mathcal{O}_\rho(M) \). Thus \( F_A = 0 \) in \( M \) and \( A \) is a flat connection. This completes the proof of Theorem 2.1 (2).

Next we prove Theorem 2.2.

By (2.5) and the Neumann condition \( i^* (+ F_A) = 0 \) on \( \partial M \), we have

\[
(2.10) \quad (n - 4) \int_M |F_A|^2 \, dx - \int_{\partial M} \langle x, \nu \rangle |F_A|^2 = 0.
\]

\( n = 4, \langle x, \nu \rangle > 0 \) on \( \partial M \) and (2.10) imply \( F_A = 0 \) on \( \partial M \). The same argument in the case of Theorem 2.1 (2) implies \( F_A = 0 \) in this case. This completes the proof. \( \square \)
3. Examples

In this section, we give an example of a principal $SU(2)$-bundle $P \to M$ such that the Dirichlet problem

\[ \begin{cases} D_A^*F_A = 0 & \text{in } M, \\ i^*A \sim 0 & \text{on } \partial M \end{cases} \]

has a non-flat solution. We also construct an example of a non-flat Yang-Mills connection for the Neumann problem when $\eta(P) \in H^2(M; \pi_1(G))$ vanishes. Our example is constructed for $M =$ cylindrical domain in dimension $4$ or, by conformal invariance, for $M = \text{annular domain} \subset \mathbb{R}^4$. The construction is based on the work of Parker \cite{9}. See also \cite{5}, \cite{16}.

Let $M = S^3 \times [0, 1]$. Identify $S^3$ with $SU(2)$. With this identification, $SU(2)$ acts $S^3$ from right and left. This action extends, in the obvious way, to an action on the trivial $SU(2)$ bundle over $M$. It is not hard to show (see \cite{9} for details) that any biinvariant connection on $M$ is gauge equivalent to one of the form

\[ A = \left( \frac{x(t) + 1}{2} \right) \sum e_i \otimes e^i, \]

where $\{e^i\}$ and $\{e_i\}$ are dual left-invariant bases of $T^*S^3$ and $TS^3$ and $x(t)$ is a real-valued function on $[0, 1]$. When this connection is restricted to the sphere $S^3 \times \{t\}$, its curvature is

\[ F_{ij} = \left( \frac{x(t)^2 - 1}{4} \right) [e_i, e_j]. \]

We can then express the Yang-Mills action in terms of $x(t)$ (again, see \cite{9}):

\[ \mathcal{YM}(A) = \frac{3}{4} \int_0^1 \frac{dx}{dt}^2 + (x^2 - 1)^2 \, dt. \]

By Palais’ symmetric criticality principle, a critical point of this functional is a Yang-Mills field (see \cite{8}, \cite{9}).

We first construct an example for the Dirichlet problem. Note that the connection (3.1) is flat on $\partial M$ if $x(0) = -1$ and $x(1) = 1$. Thus we seek critical points of $\mathcal{YM}$ under these boundary conditions. But these are easily found: direct methods in the calculus of variations show that the functional (3.3) has a minimizer on the Hilbert space

\[ H^1 = \{ x \in L^2([0, 1]) : x' \in L^2([0, 1]), x(0) = -1, x(1) = 1 \}. \]

This minimizer is a smooth solution of $x'' = x(x^2 - 1)$ with $x(0) = -1$ and $x(1) = 1$.

We claim that the connection $A$ corresponding to this $x$ is a non-flat connection with flat boundary value. To prove this, we only need to show that $A$ is not a flat connection. Suppose $A$ is flat; then $\frac{dA}{dt} = 0$ and $x^2 = 1$. But these contradict the boundary conditions $x(0) = -1$ and $x(1) = 1$. Thus we complete the proof of our claim.

Next we construct an example for the Neumann problem. Notations are the same for the Dirichlet case. First, note that $\eta(P) = 0$ since $H^2(M; \pi_1(SU(2))) = 0$. Since the curvature of $A$ is given by $F_A = dt \wedge \frac{dA}{dt} + F_A^3$, where $F_A^3$ is the curvature of the connection $A(t)$ on $S^3 \times \{t\}$, the Neumann condition is equivalent to the condition $\frac{dA}{dt}(0) = \frac{dA}{dt}(1) = 0$. Thus we need to find a solution of the equation $\frac{dx}{dt^2} = x(x^2 - 1)$ with $\frac{dx}{dt}(0) = \frac{dx}{dt}(1) = 0$. This has a trivial solution $x \equiv 0$. The
connection $A$ corresponding to this trivial solution is a solution for the Neumann problem and this is not flat since we have $YM(A) = 3/4 \neq 0$. This completes the construction for the Neumann problem.

**Remark 3.1.** (1) The above construction is related to the construction of non-self-dual Yang-Mills connections over $S^3 \times S^1$ or $S^4$. See [5], [9] and [16].

Our above examples give the first examples of non-minimal Yang-Mills connections for the Dirichlet problem and for the Neumann problem with prescribed class $\eta(P)$ (in our case $\eta(P) = 0$), since flat connection is the only minimal solution for the Dirichlet problem with flat boundary condition (see [4]) and for the Neumann problem with prescribed class $\eta(P) = 0$.

(2) It is obvious from (3.1) that the connections constructed above are irreducible connections.

From Theorem 2.1, Theorem 2.2 and these examples, we can conclude that the existence of non-flat Yang-Mills connections for the Dirichlet and Neumann problems depends on the geometry of $M$.

It would be of interest to investigate the effect of the geometry (or topology) of $M$ for the existence of the solutions to boundary value problems of Yang-Mills connections. Such relations are established for the existence problem of (anti-)self-dual connections over closed 4-manifolds, see [12] and [13]. For similar problems related to Yamabe equation and other semi-linear elliptic equations, see [2] and [3].

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**References**


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17. H. Wente, *The differential equation $\Delta x = 2H x_u \wedge x_v$ with vanishing boundary values*, Proc. Amer. Math. Soc. 50 (1975), 131–137. MR 51:10871

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