PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 6, June 1997, Pages 1585–1588
S 0002-9939(97)03809-4

FREE QUOTIENTS OF $SL_2(R[x])$

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(Communicated by Ronald M. Solomon)

ABSTRACT. It is shown that if R is an integral domain which is not a field, and $U_2(R[x])$ is the subgroup of $SL_2(R[x])$ generated by all unipotent elements, then the quotient group $SL_2(R[x])/U_2(R[x])$ has a free quotient of infinite rank.

1. Introduction

In a recent article [2], Grunewald, Mennicke and Vaserstein prove that the groups $SL_2(\mathbf{Z}[x])$ and $SL_2(K[x,y])$, where K is a finite field, can be mapped homomorphically onto a free group of an arbitrary finite rank and that, in addition, the homomorphism can be chosen so that all unipotent elements belong to its kernel. The aim of this note is to strengthen and generalize this result, at the same time providing a proof that is considerably shorter and more elementary. For background information and references we refer the reader to [2].

We will consider commutative integral domains only; for every such R, let R° denote the subset of all elements which are neither zero nor a unit, and let $\kappa(R)$ denote the product $\aleph_0|P|$, where P is a maximal subset of non-associate elements of R°. Note that $\kappa(R)$ is zero if R is a field, and is infinite otherwise. Let $U_2(R[x])$ denote the subgroup generated by all unipotent elements of $SL_2(R[x])$.

Theorem. *For every integral domain R, the group $SL_2(R[x])/U_2(R[x])$ has a free quotient of rank $\kappa(R)$.*

We can actually describe a free basis of a subgroup of $SL_2(R[x])$ which maps isomorphically onto a free quotient as in the above theorem. It consists of the matrices

$$h_{p,k} = \begin{bmatrix} 1+px^k & x^{3k} \\ p^3 & 1-px^k+p^2x^{2k} \end{bmatrix},$$

where k is a positive integer and $p \in P$.

Received by the editors October 31, 1995.
1991 *Mathematics Subject Classification.* Primary 20H25, 20E08.
The first author was partially supported by a grant from Science Fund of Serbia.
The second author's research was supported by a grant from NSERC Canada.

2. The proof

We will assume that the reader is familiar with the Bass-Serre theory of groups acting on trees as described in [5] or [1]. For example, the first five sections of the book by Dicks and Dunwoody [1] contain all the results we will use in this paper, including Nagao's theorem [4].

Let Q be the field of fractions of R and $G = SL_2(Q[x])$. Nagao's theorem says that $G = A *_C B$, where $A = SL_2(Q)$, and B and C are the upper triangular subgroups of G and A respectively. Let T denote the G-tree associated with this free product with amalgamation. The disjoint union of the set of cosets G/A and G/B is the vertex set of T, and the set of cosets G/C is the edge set of T. The edge gC connects the vertices gA and gB, and the left action of G on T is obvious.

Let us use the shorter notation H for the group $SL_2(R[x])$. This group acts on the tree T. The quotient $H\backslash T$ is a connected graph which we denote by X. By the Structure Theorem for groups acting on trees ([1], Theorem 4.1), H is the fundamental group of a graph of groups based on X and we have $H/N \cong \pi_1(X)$, where N denotes the (normal) subgroup of H generated by the vertex stabilizers. We will show that $\pi_1(X)$ has rank at least $\kappa(R)$, but let us first observe that N contains the group $U_2(R[x])$, so that, in fact, we have a homomorphism from $H/U_2(R[x])$ onto $\pi_1(X)$. Indeed, every unipotent matrix over a principal ideal domain is conjugate to an upper triangular matrix; since $Q[x]$ is a principal ideal domain, it follows that every unipotent element of H is conjugate within G to an element of B, and so fixes a vertex of T.

To obtain the required bound for the rank of $\pi_1(X)$, it will suffice to consider only the edges of T incident with the vertex B; their projections in X will span a subgraph whose fundamental group is already large enough.

The edges of T starting at the vertex B are all of the form gC, where $g \in B$. We will use the following notation for elementary matrices:

$$E(\alpha) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}.$$

For every $p \in P$ and every positive integer k we define

$$e_{p,k} = E(\alpha_{p,k})C \quad \text{and} \quad e'_{p,k} = E(\beta_{p,k})C,$$

where

$$\alpha_{p,k} = \frac{x^k}{p^2} - \frac{x^{2k}}{p} \quad \text{and} \quad \beta_{p,k} = \frac{x^k}{p^2}.$$

The edge $e_{p,k}$ connects the vertex B with the vertex $v_{p,k} = E(\alpha_{p,k})A$, and similarly the edge $e'_{p,k}$ connects B with $v'_{p,k} = E(\beta_{p,k})A$. We will prove that:

(1) $v'_{p,k} = h_{p,k}v_{p,k}$, where $h_{p,k}$ is as in the Introduction.

(2) No edge $e_{p,k}$ is H-equivalent to any edge $e'_{q,l}$, or to any other edge $e_{q,l}$.

For (1), we need to prove that $E(\beta_{p,k})A = h_{p,k}E(\alpha_{p,k})A$, which amounts to exhibiting an $a \in A$ such that $E(\beta_{p,k})aE(-\alpha_{p,k}) = h_{p,k}$. It is easy to check that we can take a to be the transpose of $E(p^3)$.

To prove (2), suppose that for some $p, q \in P$, $h \in H$, and positive integers k, l we have $e_{q,l} = he_{p,k}$ or $e'_{q,l} = he_{p,k}$. Then $E(\gamma_{q,l})cE(-\alpha_{p,k}) = h$ for some $c \in C$, where $\gamma_{q,l}$ stands for either $\alpha_{q,l}$ or $\beta_{q,l}$. Writing

$$c = \begin{bmatrix} u & r \\ 0 & u^{-1} \end{bmatrix},$$

where $u, r \in Q$, we have

$$E(\gamma_{q,l})cE(-\alpha_{p,k}) = \begin{bmatrix} u & -u\alpha_{p,k} + u^{-1}\gamma_{q,l} + r \\ 0 & u^{-1} \end{bmatrix} \in H.$$

It follows that u is a unit of R and (by letting $x = 0$) that $r \in R$. Therefore, the polynomial

$$u^{-1}\gamma_{q,l} - u\alpha_{p,k}$$

belongs to $R[x]$. Looking at the highest term we see immediately that we must have $k = l$ and $\gamma_{q,l} = \alpha_{q,k}$, in which case the coefficient of the highest term is $u/p - u^{-1}/q$. It is easy to check that this is not an element of R unless p and q are associate, which in our case means $p = q$.

It follows from (1) and (2) that the projections of the two edges $e_{p,k}$ and $e'_{p,k}$ constitute a "digon" in X, and also that the removal of the projections of all edges $e_{p,k}$ from X results in a graph that is still connected. This is enough to prove our theorem.

To obtain a free retract of H of rank $\kappa(R)$, we note first that there is a possibility that some edges $e'_{p,k}, e'_{q,l}$ are H-equivalent. To account for that, we choose a maximal subset Z of pairwise non-H-equivalent elements of $\{e'_{p,k} | p \in P,\ k \geq 1\}$ and then for every pair p, k we take $b_{p,k} \in H$ such that $b_{p,k}e'_{p,k} \in Z$. Using the language of [1], we see now that there is a fundamental H-transversal Y for T which contains the edges of $Z \cup \{e_{p,k} | p \in P,\ k \geq 1\}$ and all vertices $v'_{p,k}$ when $e'_{p,k} \in Z$, and it does not contain any vertex $v_{p,k}$. Denoting $t_{p,k} = b_{p,k}h_{p,k}$, we have that, for every p, k, the vertex $t_{p,k}v_{p,k}$ belongs to Y. Let S be the set of all $b_{p,k} \neq 1$. The Structure Theorem 4.1 of [1] and considerations around it imply now that the set $\{t_{p,k} | p \in P, k \geq 1\} \cup S$ is a basis of a free subgroup of H that injects into a free factor of the quotient H/N. The same is then clearly true for the set $\{h_{p,k} | p \in P,\ k \geq 1\}$.

3. Remarks

1. Mason uses a similar approach in [3] to study free quotients of normal subgroups of $SL_2(K[x])$, where K is a field.

2. The cardinality of the basis of the free quotient given in our theorem is clearly the best possible if R is countable, and also if $R = S[y]$ for some domain S. We do not know of any example where our cardinality result is not the best possible.

3. The free set $\{h_{p,k} | k \geq 1\}$ can be simultaneously conjugated within $SL_2(Q[x])$ into the elementary subgroup $E_2(R[x])$:

$$\begin{bmatrix} p^3 & 0 \\ 0 & 1 \end{bmatrix} h_{p,k} \begin{bmatrix} p^{-3} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + px^k & p^3x^{3k} \\ 1 & 1 - px^k + p^2x^{2k} \end{bmatrix}.$$

4. Computations like those used to prove (1) and (2) above show that for every $\alpha, \beta \in xQ[x]$ we have:

(i) The edges $E(\alpha)C$ and $E(\beta)C$ are H-equivalent if and only if there exists a unit u of R such that $\alpha - u^2\beta \in R[x]$.

(ii) The vertices $E(\alpha)A$ and $E(\beta)A$ are H-equivalent if and only if there exists a unimodular vector $(r, s) \in R \oplus R$ such that the polynomials $r\alpha, r\beta$, and $\alpha - s^2\beta + rs\alpha\beta$ all belong to $R[x]$.

This explains how we looked for digons in the graph X.

5. We do not know if $N = U_2(R[x])$ even in the simplest case $R = \mathbf{Z}$.

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