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A NOTE ON THE PATH HOLOMORPHY OF RANDOM FUNCTIONS HOLOMORPHIC IN MEAN

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Abstract. Every random function holomorphic in mean on an open subset of the complex field is equivalent to a random function with almost all its paths holomorphic on D.

Throughout the paper, $(\Omega, \Sigma, \mathbb{P})$ denotes a complete probability space and D an open subset of the complex field.

A map $f\colon D \times \Omega \to \mathbb{C}$ is said to be a (*first-order*) *random function* on D if, for each $z \in D$, the map $\omega \mapsto f(z,\omega)$ lies in $\mathcal{L}_1(\mathbb{P})$, the linear space of all *first-order random variables*. For every fixed $\omega \in \Omega$, the function $z \mapsto f(z,\omega)$ from D into $\mathbb{C}$ is called a *path* of f. Two random functions f and g on D are said to be *equivalent*, and we denote it by $f \equiv g$, if $f(z,\omega) = g(z,\omega)$ almost surely for every $z \in D$. Given $\xi \in \mathcal{L}_1(\mathbb{P})$, $[\xi]$ denotes the equivalence class of ξ for the usual almost surely identification. The space $L_1(\mathbb{P}) = \{[\xi]\colon \xi \in \mathcal{L}_1(\mathbb{P})\}$ becomes a complex Banach space with the norm $\|[\xi]\|_1 = \int_\Omega |\xi| d\mathbb{P}$.

A random function f on D is said to be *holomorphic in mean on D* if, for every $z_0 \in D$, the quotient $\frac{f(z,\cdot)-f(z_0,\cdot)}{z-z_0}$ has a limit in mean as z approaches z_0, equivalently the function $z \mapsto [f(z,\cdot)]$ from D into the complex Banach space $L_1(\mathbb{P})$ is holomorphic in the traditional sense. For a full discussion of holomorphic vector-valued functions the reader is referred to [1, Section 3.2]. It should be pointed out that a random function f is holomorphic in mean on D if, and only if, for every $\xi \in \mathcal{L}_\infty(\mathbb{P})$ the complex valued function $z \mapsto \int_\Omega f(z,\omega)\xi(\omega) d\mathbb{P}$ is holomorphic on D, where $\mathcal{L}_\infty(\mathbb{P})$ is the linear space of all essentially bounded random variables (see [1, Definition 3.10.1 and Proposition 3.10.1]).

The purpose of this paper is to prove the following surprising result.

Theorem. *Let f be a random function on D. Then the following conditions are equivalent:*

1. *f is holomorphic in mean on D.*
2. *f is equivalent to a random function g continuous in mean with almost all its paths holomorphic on D.*

We note that there are random functions holomorphic in mean which have no holomorphic paths (see Example 1) and we emphasize that there exists a random function f continuous in mean, but such that there is no random function g that is equivalent to f and has a non-negligible set of continuous paths (see Example 2).

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Example 1. Let $D = \{x + iy \in \mathbb{C}\colon 0 < x, y < 1\}$ and consider Ω as D endowed with the Lebesgue measure. The function $f\colon D \times \Omega \to \mathbb{C}$ given by $f(z, \omega) = 1$ if $z = \omega$ and $f(z, \omega) = 0$ otherwise, is holomorphic in mean on D and has no holomorphic paths.

Example 2. Consider the interval $[0, 1]$ endowed with the Lebesgue measure and the sequence $\{\xi_n\}$ of random variables given by $\xi_n(\omega) = 1$ if $2^{-k}j \leq \omega \leq 2^{-k}(j+1)$ and $\xi_n(\omega) = 0$ otherwise, where $n = 2^k + j$, $0 \leq k$, $0 \leq j < 2^k$. Define the random function f on the open unit disc D by $f(z, \omega) = n^2(n+1)^2(\frac{1}{n} - |z|)(|z| - \frac{1}{n+1})\xi_n(\omega)$ if $\frac{1}{n+1} \leq |z| \leq \frac{1}{n}$, $n \in \mathbb{N}$, and $f(0, \omega) = 0$, which is obviously continuous in mean on $D\backslash\{0\}$. Given a sequence $\{z_n\}$ in $D\backslash\{0\}$ converging to zero we have $\int_0^1 |f(z_n, \omega)| d\omega \leq \frac{1}{4}\|\xi_{k_n}\|_1$, where k_n satisfies $\frac{1}{k_n+1} \leq |z_n| \leq \frac{1}{k_n}$. Since $\{\|\xi_{k_n}\|_1\}$ converges to zero we conclude that f is continuous in mean at zero. Now we note that $f(\frac{2n+1}{2n(n+1)}, \omega) = \frac{1}{4}\xi_n(\omega)$ for all $n \in \mathbb{N}$ and $\omega \in [0, 1]$. Since the sequence $\{\xi_n\}$ converges nowhere on $[0, 1]$ it may be concluded that almost every path of any random function equivalent to f is not continuous at zero.

Given a random function continuous in mean on D and γ a piecewise smooth curve in D, parametrized say in the interval $[\alpha, \beta]$, the function $t \mapsto [f(\gamma(t), \cdot)]\gamma'(t)$ from $[\alpha, \beta]$ into $L_1(\mathbb{P})$ is Riemann integrable. Let us denote $\int_\gamma [f(z, \cdot)] dz = \int_\alpha^\beta [f(\gamma(t), \cdot)]\gamma'(t)$.

Lemma 1. *Let f be a random function continuous in mean on D and assume that almost all its paths are continuous on D. If γ is a piecewise smooth curve in D, then the function $\omega \mapsto \int_\gamma f(z, \omega)\, dz$ lies in the equivalence class $\int_\gamma [f(z, \cdot)] dz$.*

Proof. There is no loss of generality in assuming γ parametrized in the interval $[0, 1]$.

By [2, Proposition 2.7.12] the sequence $\{[\frac{1}{n}\sum_{j=1}^n f(\gamma(\frac{j}{n}), \cdot)\gamma'(\frac{j}{n})]\}$ converges in mean to $\int_\gamma [f(z, \cdot)] dz$. From [2, Proposition 2.3.10] it follows that there exists a strictly increasing sequence $\{n_k\}$ of natural numbers such that the sequence $\{\frac{1}{n_k}\sum_{j=1}^{n_k} f(\gamma(\frac{j}{n_k}), \cdot)\gamma'(\frac{j}{n_k})\}$ converges almost surely to a function in $\mathcal{L}_1(\mathbb{P})$. We note that, for almost every $\omega \in \Omega$, $\{\frac{1}{n_k}\sum_{j=1}^{n_k} f(\gamma(\frac{j}{n_k}), \omega)\gamma'(\frac{j}{n_k})\}$ converges to $\int_\gamma f(z, \omega) dz$ and therefore, from [2, Corollary 2.3.12], it may be concluded that the function $\omega \mapsto \int_\gamma f(z, \omega) dz$ lies in $\int_\gamma [f(z, \cdot)] dz$. $\square$

Lemma 2. *Let $\sum \xi_n(z - z_0)^n$ a power series with first-order random coefficients and a non-zero radius of convergence in mean, say R. Then there exists a negligible set Δ such that, for each $\omega \in \Omega\backslash\Delta$, the radius of convergence of the power series $\sum \xi_n(\omega)(z - z_0)^n$ is at least R.*

Proof. Consider $0 < r < R$. By [1, Theorem 3.11.4] the series $\sum \int_\Omega |\xi_n| r^n d\mathbb{P}$ converges and therefore [2, Proposition 2.5.1] there exists a negligible set Δ_r such that, for every $\omega \in \Omega\backslash\Delta_r$, the series $\sum \xi_n(\omega) r^n$ converges (absolutely) to a function in $\mathcal{L}_1(\mathbb{P})$. Moreover, for all $\omega \in \Omega\backslash\Delta_r$ and $|z - z_0| \leq r$ the series $\sum \xi_n(\omega)(z - z_0)^n$ converges. Choose a sequence $\{r_n\} \to R$ with $0 < r_n < R$ $\forall n \in \mathbb{N}$ and let us denote by Δ the negligible set given by $\Delta = \bigcup_{n=1}^\infty \Delta_{r_n}$. It is a simple matter to show that, for all $\omega \in \Omega\backslash\Delta$ and $|z - z_0| < R$ the series $\sum \xi_n(\omega)(z - z_0)^n$ converges, which completes the proof. $\square$

Lemma 3. *Let f and g be random functions holomorphic in mean on D and assume that f and g have almost all its paths holomorphic on D. If $f \equiv g$, then* $\mathbb{P}[f(\cdot,\omega) = g(\cdot,\omega)] = 1$.

Proof. Let $\Lambda \in \Sigma$ with $\mathbb{P}[\Lambda] = 1$ and such that the paths $f(\cdot,\omega)$ and $g(\cdot,\omega)$ are holomorphic on D whenever ω lies in Λ.

Write $D = \bigcup_{n=1}^{\infty} D_n$ for a sequence $\{D_n\}$ of pairwise disjoint connected open subsets. Fix $n \in \mathbb{N}$ and let $\Delta_n = \{\omega \in \Omega \colon f(z,\omega) = g(z,\omega)\ \forall z \in D_n\}$. Choose a sequence $\{z_k\}$ in D_n converging to $z_0 \in D$. From the uniqueness theorem [1, Theorem 3.11.5] it follows that, for a given $\omega \in \Lambda$, $f(z,\omega) = g(z,\omega)\ \forall z \in D_n$ if, and only if, $f(z_k,\omega) = g(z_k,\omega)\ \forall k \in \mathbb{N} \cup \{0\}$. Thus we have

$$\Delta_n \cap \Lambda = \bigcap_{k=0}^{\infty} \{\omega \in \Lambda \colon f(z_k,\omega) = g(z_k,\omega)\},$$

which proves that Δ_n is measurable with $\mathbb{P}[\Delta_n] = 1$. Therefore $\mathbb{P}[f(\cdot,\omega) = g(\cdot,\omega)] = 1$, since $\{\omega \in \Omega \colon f(\cdot,\omega) = g(\cdot,\omega)\} = \bigcap_{n=1}^{\infty} \Delta_n$. □

Proof of the Theorem. Assume the assertion 1 holds. If $D = \mathbb{C}$, then $[f(z,\cdot)]$ can be expanded on $\mathbb{C}$ as the sum in mean of a power series $\sum [\xi_n] z^n$ with $\xi_n \in \mathcal{L}_1(\mathbb{P})\ \forall n \in \mathbb{N} \cup \{0\}$. The series $\sum \xi_n z^n$ satisfies the requirements in Lemma 2. If Δ is the negligible set given by that lemma, then we define $g(z,\omega) = \sum_{n=0}^{\infty} \xi_n(\omega) z^n\ \forall (z,\omega) \in \mathbb{C} \times (\Omega \backslash \Delta)$ and $g(z,\omega) = 0$ otherwise. The function g clearly satisfies our requirements in assertion 2. If $D \neq \mathbb{C}$, then we consider the countable set $B = \{p + qi \in D \colon p, q \in \mathbb{Q}\}$. For each $b \in B$, set $r_b = \inf\{|z - b| \colon z \notin D\}$ and write $D_b = \{z \in \mathbb{C} \colon |z - b| < r_b\}$. For every $b \in B$ we can write $[f(z,\cdot)]$ on D_b as the sum in mean of a power series $\sum [\xi_n^{(b)}](z-b)^n$ with $\xi_n^{(b)} \in \mathcal{L}_1(\mathbb{P})\ \forall n \in \mathbb{N} \cup \{0\}$. The series $\sum \xi_n^{(b)}(z-b)^n$ satisfies the requirements in Lemma 2. Let us denote by Δ_b the negligible set given by that lemma and let $g_b(z,\omega) = \sum \xi_n^{(b)}(\omega)(z-b)^n\ \forall (z,\omega) \in D_b \times (\Omega \backslash \Delta_b)$ and $g(z,\omega) = 0$ otherwise. Note that g_b is holomorphic in mean on D_b and all its paths are holomorphic on D_b. Given $b, c \in B$ with $D_b \cap D_c \neq \varnothing$, g_b and g_c are clearly equivalent on $D_b \cap D_c$ and Lemma 3 gives a negligible set $\Delta_{b,c}$ such that $g_b(\cdot,\omega) = g_c(\cdot,\omega)\ \forall \omega \in \Omega \backslash \Delta_{b,c}$. Finally we take $\Delta = \bigcup_{b,c \in B} \Delta_{b,c}$ and we define the function g on $D \times \Omega$ by $g(z,\omega) = g_b(z,\omega)$ if $(z,\omega) \in D_b \times \Omega \backslash \Delta$ and $g(z,\omega) = 0$ otherwise. Then g is a well-defined first-order random function on D with holomorphic paths. Moreover, it is clear that $f \equiv g$.

Conversely, suppose that assertion 2 holds. Note that f is continuous in mean on D. In order to prove the holomorphy in mean of f we can assume that D is an open disc. Consider $\xi \in \mathcal{L}_\infty(\mathbb{P})$. For all $z, w \in D$ we have

$$\left| \int_\Omega f(z,\omega)\xi(\omega) d\mathbb{P} - \int_\Omega f(w,\omega)\xi(\omega) d\mathbb{P} \right| \leq \|\xi\|_\infty \|f(z,\cdot) - f(w,\cdot)\|_1,$$

which shows the continuity on D of the function $\int_\Omega f(\cdot,\omega)\xi(\omega) d\mathbb{P}$. Let γ be a piecewise smooth closed curve in D. Then $\int_\gamma g(z,\omega) dz = 0$ for almost every $\omega \in \Omega$, since almost every path function of g is holomorphic on D. From Lemma 1 it follows that $\int_\gamma [g(z,\cdot)] dz = 0$ and finally we note that $\int_\gamma [f(z,\cdot)] dz = \int_\gamma [g(z,\cdot)] dz$. On account of the continuity of the linear functional $\zeta \mapsto \int_\Omega \zeta \xi d\mathbb{P}$ on $L_1(\mathbb{P})$, [2, Proposition 2.3.7(5)] shows that

$$0 = \int_\Omega \left(\int_\gamma [f(z,\cdot)] dz \right) \xi \, d\mathbb{P} = \int_\gamma \left(\int_\Omega f(z,\omega)\xi(\omega) d\mathbb{P} \right) dz.$$

According to Morera's Theorem, every function $\int_\Omega f(\cdot,\omega)\xi(\omega)d\mathbb{P}$ is holomorphic on D and we conclude that f is holomorphic in mean on D. $\square$

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