A NOTE ON THE PATH HOLOMORPHY OF RANDOM FUNCTIONS HOLOMORPHIC IN MEAN

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Abstract. Every random function holomorphic in mean on an open subset of the complex field is equivalent to a random function with almost all its paths holomorphic on D.

Throughout the paper, \((\Omega, \Sigma, \mathbb{P})\) denotes a complete probability space and \(D\) an open subset of the complex field.

A map \(f : D \times \Omega \to \mathbb{C}\) is said to be a (first-order) random function on \(D\) if, for each \(z \in D\), the map \(\omega \mapsto f(z, \omega)\) lies in \(L_1(\mathbb{P})\), the linear space of all first-order random variables. For every fixed \(\omega \in \Omega\), the function \(z \mapsto f(z, \omega)\) from \(D\) into \(\mathbb{C}\) is called a path of \(f\). Two random functions \(f\) and \(g\) on \(D\) are said to be equivalent, and we denote it by \(f \equiv g\), if \(f(z, \omega) = g(z, \omega)\) almost surely for every \(z \in D\).

Given \(\xi \in L_1(\mathbb{P})\), \([\xi]\) denotes the equivalence class of \(\xi\) for the usual almost surely identification. The space \(L_1(\mathbb{P}) = \{[\xi] : \xi \in L_1(\mathbb{P})\}\) becomes a complex Banach space with the norm \(\|\xi\|_1 = \int_\Omega |\xi|d\mathbb{P}\).

A random function \(f\) on \(D\) is said to be holomorphic in mean on \(D\) if, for every \(z_0 \in D\), the quotient \(\frac{f(z, \cdot) - f(z_0, \cdot)}{z - z_0}\) has a limit in mean as \(z\) approaches \(z_0\), equivalently the function \(z \mapsto [f(z, \cdot)]\) from \(D\) into the complex Banach space \(L_1(\mathbb{P})\) is holomorphic in the traditional sense. For a full discussion of holomorphic vector-valued functions the reader is referred to [1, Section 3.2]. It should be pointed out that a random function \(f\) is holomorphic in mean on \(D\) if, and only if, for every \(\xi \in L_\infty(\mathbb{P})\), the complex valued function \(z \mapsto \int_\Omega f(z, \omega)\xi(\omega)d\mathbb{P}\) is holomorphic on \(D\), where \(L_\infty(\mathbb{P})\) is the linear space of all essentially bounded random variables (see [1, Definition 3.10.1 and Proposition 3.10.1]).

The purpose of this paper is to prove the following surprising result.

Theorem. Let \(f\) be a random function on \(D\). Then the following conditions are equivalent:

1. \(f\) is holomorphic in mean on \(D\).
2. \(f\) is equivalent to a random function \(g\) continuous in mean with almost all its paths holomorphic on \(D\).

We note that there are random functions holomorphic in mean which have no holomorphic paths (see Example 1) and we emphasize that there exists a random function \(f\) continuous in mean, but such that there is no random function \(g\) that is equivalent to \(f\) and has a non-negligible set of continuous paths (see Example 2).
Example 1. Let $D = \{ x + iy \in \mathbb{C} : 0 < x, y < 1 \}$ and consider $\Omega$ as $D$ endowed with the Lebesgue measure. The function $f : D \times \Omega \to \mathbb{C}$ given by $f(z, \omega) = 1$ if $z = \omega$ and $f(z, \omega) = 0$ otherwise, is holomorphic in mean on $D$ and has no holomorphic paths.

Example 2. Consider the interval $[0, 1]$ endowed with the Lebesgue measure and the sequence $\{\xi_n\}$ of random variables given by $\xi_n(\omega) = 1$ if $2^{-k} < j \leq 2^{-k}(j+1)$ and $\xi_n(\omega) = 0$ otherwise, where $n = 2^k + j$, $0 \leq k$, $0 \leq j < 2^k$. Define the random function $f$ on the open unit disc $D$ by $f(z, \omega) = n^2(n+1)^2(\frac{1}{n} - |z|)(|z| - \frac{1}{n+1})\xi_n(\omega)$ if $\frac{1}{n+1} \leq |z| \leq \frac{1}{n}$, $n \in \mathbb{N}$, and $f(0, \omega) = 0$, which is obviously continuous in mean on $D\setminus \{0\}$. Given a sequence $\{z_n\}$ in $D\setminus \{0\}$ converging to zero we have $\int_0^1 |f(z_n, \omega)|d\omega \leq \frac{1}{2}||\xi_n||_1$, where $k_n$ satisfies $\frac{1}{n+1} \leq |z_n| \leq \frac{1}{n}$. Since $||\xi_n||_1$ converges to zero we conclude that $f$ is continuous in mean at zero. Now we note that $f(\frac{2n+1}{2n(n+1)}, \omega) = \frac{1}{2}\xi_n(\omega)$ for all $n \in \mathbb{N}$ and $\omega \in [0, 1]$. Since the sequence $\{\xi_n\}$ converges nowhere on $[0, 1]$ it may be concluded that almost every path of any random function equivalent to $f$ is not continuous at zero.

Given a random function continuous in mean on $D$ and $\gamma$ a piecewise smooth curve in $D$, parametrized say in the interval $[\alpha, \beta]$, the function $t \mapsto [f(\gamma(t), \cdot)]\gamma'(t)$ from $[\alpha, \beta]$ into $L_1(\Omega)$ is Riemann integrable. Let us denote $\int_\gamma [f(\gamma(t), \cdot)]dz = \int_\alpha^\beta [f(\gamma(t), \cdot)]\gamma'(t)$.

Lemma 1. Let $f$ be a random function continuous in mean on $D$ and assume that almost all its paths are continuous on $D$. If $\gamma$ is a piecewise smooth curve in $D$, then the function $\omega \mapsto \int_\gamma f(z, \omega)dz$ lies in the equivalence class $\int_\gamma [f(z, \cdot)]dz$.

Proof. There is no loss of generality in assuming $\gamma$ parametrized in the interval $[0, 1]$.

By [2, Proposition 2.7.12] the sequence $\{\frac{1}{n} \sum_{j=1}^n f(\gamma(\frac{j}{n}), \cdot)\gamma'(\frac{j}{n})\}$ converges in mean to $\int_\gamma f(z, \cdot)dz$. From [2, Proposition 2.3.10] it follows that there exists a strictly increasing sequence $\{n_k\}$ of natural numbers such that the sequence $\{\frac{1}{n_k} \sum_{j=1}^{n_k} f(\gamma(\frac{j}{n_k}), \cdot)\gamma'(\frac{j}{n_k})\}$ converges almost surely to a function in $L_1(\Omega)$. We note that, for almost every $\omega \in \Omega$, $\frac{1}{n_k} \sum_{j=1}^{n_k} f(\gamma(\frac{j}{n_k}), \omega)\gamma'(\frac{j}{n_k})$ converges to $\int_\gamma f(z, \omega)dz$ and therefore, from [2, Corollary 2.3.12], it may be concluded that the function $\omega \mapsto \int_\gamma f(z, \omega)dz$ lies in $\int_\gamma [f(z, \cdot)]dz$.

Lemma 2. Let $\sum \xi_n(z - z_0)^n$ a power series with first-order random coefficients and a non-zero radius of convergence in mean, say $R$. Then there exists a negligible set $\Delta$ such that, for each $\omega \in \Omega \setminus \Delta$, the radius of convergence of the power series $\sum \xi_n(\omega)(z - z_0)^n$ is at least $R$.

Proof. Consider $0 < r < R$. By [1, Theorem 3.11.1] the series $\sum \int_{\Omega} |\xi_n|^rd\mathbb{P}$ converges and therefore [2, Proposition 2.5.1] there exists a negligible set $\Delta_r$ such that, for every $\omega \in \Omega \setminus \Delta_r$, the series $\sum \xi_n(\omega)r^n$ converges (absolutely) to a function in $L_1(\Omega)$. Moreover, for all $\omega \in \Omega \setminus \Delta_r$ and $|z - z_0| \leq r$ the series $\sum \xi_n(\omega)(z - z_0)^n$ converges. Choose a sequence $\{r_n\} \to R$ with $0 < r_n < R \forall n \in \mathbb{N}$ and let us denote by $\Delta$ the negligible set given by $\Delta = \bigcup_{n=1}^\infty \Delta_{r_n}$. It is a simple matter to show that, for all $\omega \in \Omega \setminus \Delta$ and $|z - z_0| < R$ the series $\sum \xi_n(\omega)(z - z_0)^n$ converges, which completes the proof.
Lemma 3. Let $f$ and $g$ be random functions holomorphic in mean on $D$ and assume that $f$ and $g$ have almost all its paths holomorphic on $D$. If $f \equiv g$, then $\mathbb{P}[f(\cdot, \omega) = g(\cdot, \omega)] = 1$.

Proof. Let $\Lambda \in \Sigma$ with $\mathbb{P}[\Lambda] = 1$ and such that the paths $f(\cdot, \omega)$ and $g(\cdot, \omega)$ are holomorphic on $D$ whenever $\omega$ lies in $\Lambda$.

Write $D = \bigcup_{n=1}^{\infty} D_n$ for a sequence $\{D_n\}$ of pairwise disjoint connected open subsets. Fix $n \in \mathbb{N}$ and let $\Delta_n = \{\omega \in \Omega : f(z, \omega) = g(z, \omega) \forall z \in D_n\}$. Choose a sequence $\{z_k\}$ in $D_n$ converging to $z_0 \in D$. From the uniqueness theorem [1, Theorem 3.11.5] it follows that, for a given $\omega \in \Lambda$, $f(z, \omega) = g(z, \omega) \forall z \in D_n$ if, and only if, $f(z_k, \omega) = g(z_k, \omega) \forall k \in \mathbb{N} \cup \{0\}$. Thus we have

$$\Delta_n \cap \Lambda = \bigcap_{k=0}^{\infty} \{\omega \in \Lambda : f(z_k, \omega) = g(z_k, \omega)\},$$

which proves that $\Delta_n$ is measurable with $\mathbb{P}[\Delta_n] = 1$. Therefore $\mathbb{P}[f(\cdot, \omega) = g(\cdot, \omega)] = 1$, since $\{\omega \in \Omega : f(\cdot, \omega) = g(\cdot, \omega)\} = \bigcap_{n=1}^{\infty} \Delta_n$. \hfill \Box

Proof of the Theorem. Assume the assertion 1 holds. If $D = \mathbb{C}$, then $[f(z, \cdot)]$ can be expanded on $\mathbb{C}$ as the sum in mean of a power series $\sum |\xi_n| z^n$ with $\xi_n \in \mathbb{C}$ for all $n \in \mathbb{N} \cup \{0\}$. The series $\sum |\xi_n| z^n$ satisfies the requirements in Lemma 2. If $\Delta$ is the negligible set given by that lemma, then we define $g(z, \omega) = \sum_{n=0}^{\infty} \xi_n(\omega) z^n \forall (z, \omega) \in \mathbb{C} \times (\Omega \setminus \Delta)$ and $g(z, \omega) = 0$ otherwise. The function $g$ clearly satisfies our requirements in assertion 2. If $D \neq \mathbb{C}$, then we consider the countable set $B = \{p + qi : p, q \in \mathbb{Q}\}$. For each $b \in B$, set $r_b = \inf \{|z-b| : z \notin D\}$ and write $D_b = \{z \in \mathbb{C} : |z-b| < r_b\}$. For every $b \in B$ we can write $[f(z, \cdot)]$ on $D_b$ as the sum in mean of a power series $\sum |\xi_n^{(b)}| (z-b)^n$ with $\xi_n^{(b)} \in \mathbb{C}$ for all $n \in \mathbb{N} \cup \{0\}$. The series $\sum |\xi_n^{(b)}| (z-b)^n$ satisfies the requirements in Lemma 2. Let us denote by $\Delta_b$ the negligible set given by that lemma and let $g_b(z, \omega) = \sum \xi_n^{(b)}(\omega) (z-b)^n \forall (z, \omega) \in D_b \times (\Omega \setminus \Delta_b)$ and $g(z, \omega) = 0$ otherwise. Note that $g_b$ is holomorphic in mean on $D_b$ and all its paths are holomorphic on $D_b$. Given $b, c \in B$ with $D_b \cap D_c \neq \emptyset$, $g_b$ and $g_c$ are clearly equivalent on $D_b \cap D_c$ and Lemma 3 gives a negligible set $\Delta_{b,c}$ such that $g_b(\cdot, \omega) = g_c(\cdot, \omega) \forall \omega \in \Omega \setminus \Delta_{b,c}$. Finally we take $\Delta = \bigcup_{b,c \in B} \Delta_{b,c}$ and we define the function $g$ on $D \times \Omega$ by $g(z, \omega) = g_b(z, \omega)$ if $(z, \omega) \in D_b \times (\Omega \setminus \Delta)$ and $g(z, \omega) = 0$ otherwise. Then $g$ is a well-defined first-order random function on $D$ with holomorphic paths. Moreover, it is clear that $f \equiv g$.

Conversely, suppose that assertion 2 holds. Note that $f$ is continuous in mean on $D$. In order to prove the holomorphy in mean of $f$ we can assume that $D$ is an open disc. Consider $\xi \in \mathcal{L}_2(\mathbb{P})$. For all $z, w \in D$ we have

$$\int_{\Omega} f(z, \cdot) \xi(\omega) d\mathbb{P} - \int_{\Omega} f(w, \cdot) \xi(\omega) d\mathbb{P} \leq \|\xi\|_2 \|f(z, \cdot) - f(w, \cdot)\|_2,$$

which shows the continuity on $D$ of the function $\int f(\cdot, \omega) \xi(d\omega)$. Let $\gamma$ be a piecewise smooth closed curve in $D$. Then $\int f(z, \omega) d\omega = 0$ for almost every $\omega \in \Omega$, since almost every path function of $g$ is holomorphic on $D$. From Lemma 1 it follows that $\int_{\gamma} [g(z, \cdot)] d\gamma = 0$ and finally we note that $\int_{\gamma} [f(z, \cdot)] d\gamma = \int_{\gamma} [g(z, \cdot)] d\gamma$. On account of the continuity of the linear functional $\zeta \mapsto \int_{\Omega} \zeta d\mathbb{P}$ on $L_2(x)\), [2, Proposition 2.3.7(5)] shows that

$$0 = \int_{\gamma} \left( \int f(z, \cdot) d\omega \right) \zeta d\mathbb{P} = \int_{\gamma} \left( \int f(z, \omega) \xi(\omega) d\mathbb{P} \right) d\omega.$$
According to Morera’s Theorem, every function $\int_{\Omega} f(\cdot, \omega) \xi(\omega) d\mathbb{P}$ is holomorphic on $D$ and we conclude that $f$ is holomorphic in mean on $D$.

References


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